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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

INVESTIGATION AND DEVELOPMENT OF METHODS OF SOLUTIONS OF MIXED-INTEGER PROGRAMMING PROBLEMS WİTH INTERVAL COEFFICIENTS

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GENERAL DESCRIPTION OF WORK

Relevance of the topic. As you know, mathematical models of many applied problems of economics and technology associated with making optimal decisions are obtained in the form of various classes of partial-integer programming problems. Since the middle of the last century, these problems have been intensively studied by various authors in order to develop methods for their solution.

Note that various classes of integer programming problems, including those with interval coefficients[1], have been considered by many scientists such as Babaev J.F., Veliev G.P., Guseinov S.Ya., Devyaterikova M.V., Emelichev V.A. ., Ferkha U., Kolokolov A.A., Kovalev M.M., Kravtsov M.K., Mamedov K.Sh., Mamedova A.G., Li W., Li W., Liu X., Li H ., Martello S., Pisinger D., Hladik M.[2]. However, all known methods have exponential complexities. Such problems in the literature are called hard-to-solve and are included in the class of NP-complete. Therefore, special methods have been developed for constructing an approximate (suboptimal) solution of various classes of partial-integer programming problems.

It should be noted that in economics there are mathematical models of such problems in which some of the variables must take integers, and the rest any values. In such models, the economic meaning (value) of the coefficients of an objective function, constraints and right-hand sides mean profits, costs and allocated common resources, respectively.

It should be especially noted that in a modern market economy, if profits, costs and allocated common resources are specific

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[1] Li, W. Generalized solutions to interval linear programmes and related necessary and sufficient optimality conditions / W. Li, X. Liu, H. Li // Optimization Methods Software, − 2015. 30(3), − p.516-530.

[2] Hladik M. On strong optimality of interval linear programming // Optimization Letters, − 2017. 11(7),−p.1459-1468.

numbers, then the corresponding model cannot adequately (equivalently) describe the process. Because in the real world, the revenues generated, the costs and the shared resources allocated should be allowed to vary within certain limits (intervals). Obviously, such models will describe the process more reliably.

Mathematical models of such problems can be represented as follows:

$$\sum_{j=1}^{n}[\underline{c}_j,\overline{c}_j]\,x_j+\sum_{j=n+1}^{N}[\underline{c}_j,\overline{c}_j]\,x_j\to\max, \tag{1}$$

$$\sum_{j=1}^{n}[\underline{a}_{ij},\overline{a}_{ij}]\,x_j+\sum_{j=n+1}^{N}[\underline{a}_{ij},\overline{a}_{ij}]\,x_j\le[\underline{b}_i,\overline{b}_i],\quad (i=\overline{1,m}), \tag{2}$$

$$0\le x_j\le d_j,\ \ (j=\overline{1,N}), \tag{3}$$

$$x_j,\text{ integer},\ (j=\overline{1,n}),\ \ (n\le N) \tag{4}$$

Here, it is assumed that $0<\underline{c}_j\le\overline{c}_j, 0\le\underline{a}_{ij}\le\overline{a}_{ij},\ 0<\underline{b}_i\le\overline{b}_i,\ d_j>0, (i=\overline{1,m};\ j=\overline{1,N})$ – given integers. More over, intervals $[\underline{c}_j,\overline{c}_j]$, $[\underline{a}_{ij},\overline{a}_{ij}]$ and $[\underline{b}_i,\overline{b}_i]$, $(i=\overline{1,m};\ j=\overline{1,N})$ mean the interval of changes in profit, costs and allocated resources, respectively. Note that problem (1) - (4) is called a partial integer programming problem with interval coefficients or an interval partial integer programming problem.

It is pertinent to note that problem (1) - (4) is a generalization of the following problems: 1) Boolean programming problems; 2) an integer programming problem; 3) an interval problem of Boolean programming; 4) an interval integer programming problem; 5) interval problem of partial-Boolean programming; 6) problems of linear programming; 7) an interval linear programming problem, etc.

As: 1) in cases, where the endpoints of the intervals coincide, $n = N$ и $d_j = 1,\ (j = \overline{1, N})$, Boolean programming problem is obtained; 2) if $n = N$ and the endpoints of the intervals coincide, an integer programming problem is obtained; 3) if $d_j = 1,\ (j = \overline{1, N})$ and $n = N$ an interval problem of Boolean programming; 4) if $n = N$ an interval integer programming problem is obtained; 5) if $d_j = 1$, $(j = \overline{1, N})$ and $n < N$ an interval problem of partial-Boolean programming is obtained; 6) if the endpoints of the intervals coincide and $n = 0$, a problem of linear programming is obtained; 7) if $n = 0$ an interval linear programming problem.

Note that the considered problem (1) - (4) belongs to the class of NP-complete, since all of the above particular cases of this problem, except for linear programming problems, are NP-complete, in other words, difficult to solve[3]. Various classes of this problem have been investigated and specific methods developed.

It should be noted that as far as we know, the problem of partial-integer programming with interval data has not yet been investigated. This may be due to the fact that they are included in the class of NP-complete, as well as to the complexity of the development of methods for their exact or approximate solution.

From the above, it follows that there is a need for research and development of methods for solving problems (1) - (4).

Therefore, the relevance of the chosen topic of the dissertation work does not raise any doubts.

Object and subject of research. In the dissertation work, the object of research and the subject of the topic are the development of methods and algorithms for finding approximate (suboptimistic and subestimistic) solutions for

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[3] Гэри, М. Вычислительные машины и труднорешаемые задачи / М. Гэри, Д. Джонсон.– Москва: Мир, –1982. – с.416.

various classes of partial-integer programming problems with interval coefficients. These types of problems are found in the objects of the economy, where it is necessary to make an optimal decision.

The main purpose of the thesis statement is:

– Introduce the concepts of optimistic, pessimistic, suboptimistic and subpessimistic solutions for the interval partial Boolean knapsack problem, the interval partial Boolean programming problem, and develop methods for constructing suboptimistic and subpessimistic solutions to these problems.

– Introduce the concepts of optimistic, pessimistic, suboptimistic and subpessimistic solutions of the partial-integer knapsack problem with interval data, the interval partial-integer programming problem and, on the basis of these, develop a method for constructing suboptimistic and subpessimistic solutions to these problems.

– To estimate the error of optimal and approximate (i.e. suboptimistic and subpessimistic) solutions, develop an algorithm for finding the upper bound of the optimum in the interval partial Boolean knapsack problem and the interval partial Boolean programming problem.

– Construct a majorizing function with respect to an objective function in an interval problem of partial Boolean programming and prove its basic properties.

– Development of an algorithm for minimizing a majorizing function to determine the upper bound of the optimistic and pessimistic values of the objective function in an interval problem of partial Boolean programming.

– Compile software for the developed algorithms and carry out comparative experiments.

Research methods. In the dissertation work, the principles of modern optimization theory, theories and methods of mathematical and discrete programming, methods of combinatorial analysis, etc. were used.

The main provisions for the defense.

1. Methods for constructing suboptimistic and subpessimistic solutions of a partially-Boolean and partially-integer knapsack prob lem with interval coefficients
2. Methods for constructing suboptimistic and subpessimistic solu tions to a partially Boolean and partially integer programming problem with interval coefficients
3. Method of nonlinearly increasing penalty for constructing suboptimistic and subpessimistic solutions to the interval problem of partially Boolean and partially integer programming
4. Using the proposed methods, conducting numerous comparative experiments on large-scale problems.
5. Finding the upper bound of the maximum value of the objective function in the interval partial Boolean knapsack problem.
6. Construction of a majorizing function with respect to an objective function in an interval problem of partial-Boolean programming and its properties
7. Algorithm for minimization of the majorizing function for opti mistic and pessimistic problems.

Scientific novelty.

– The concepts of optimistic, pessimistic, suboptimistic and subpessimistic solutions for the partial Boolean knapsack problem with interval coefficients and the interval problem of partial Boolean programming have been introduced, and algorithms have been developed for constructing suboptimistic and subpessimistic solutions to these problems.

– Similarly, the above definitions were introduced for the interval problem of partial-integer programming and algorithms for constructing suboptimistic and subpessimistic solutions were developed.

– To estimate the errors of approximate solutions from the optimal one, a majorizing function was constructed with respect to the maximum value of the objective function in an interval problem of partial Boolean programming, its basic properties were proved, and an algorithm of the fastest coordinate descent type was developed to minimize this function.

– Based on the proposed methods, a complex of programs was compiled and numerous computational experiments were carried out.

The theoretical and practical significance of the work. The methods, algorithms and software developed in the dissertation work can be used to solve various applied problems of an economic and technical nature. In particular, in some industrial areas, where individual and non-piece goods are produced, the models and methods discussed in the dissertation can be applied.

In addition, using the software systems presented in the application, you can successfully solve this type of various problems. It should be noted that with the use of these software tools, it is possible to determine not only approximate solutions of the corresponding problems, but also an assessment of their proximity to the optimal one.

Approbation and application. The main results obtained in the dissertation were reported and discussed in the following Local, International conferences and seminars:

Əmir Şamil oğlu Həbibzadənin anadan olmasının 100-cü ildönümünə həsr olunmuş "Funksional Analiz və Onun Tətbiqləri" adlı konf.(2016, Bakı); "Riyaziyyatın Tətbiqi Məsələləri və Yeni İnformasiya Texnologiyları" adlı III Respublika Elmi Konf.(15-16 dekabr 2016, Bakı); Riyaziyyatın Nəzəri və Tətbiqi Problemləri

Beynəlxalq Elmi Konf. (25-26 may 2017, Bakı); 6-th International Conference on Control and Optimization with Industrial Applications (11-13 july, 2018, Baku)(report was certified); Актуальные направления научных исследований XXI века: "Теория и практика" (2019, Воронеж); 7-th International Conference on Control and Optimization with Industrial Applications (26-28 august, 2020, Baku); Riyaziyyatin Fundamental Problemləri və Intellektual Texnologiyalarin Təhsildə Tətbiqi Respublika Elmi Konf. (2020, Sumaqyıt.); in Scientific seminars of lab."Models and Mehtods of Discrete Optimizization" of Institute of Control Systems of ANAS.

Publications. On the topic of the dissertation work, 16 scientific works were published, of which 9 are articles, 5 of which were published in foreign countries, 7 articles were published in the materials and abstracts of international conferences.

The name of the institution where the dissertation work was performed. The dissertation work was carried out at the Institute of Management Systems of the National Academy of Sciences of Azerbaijan.

T**he structure and volume of the dissertation.** The dissertation work consists of an introduction, 3 chapters, a bibliography of 198 references and Appendices. The total volume of the dissertation, including 36 tables is 164 pages (214190 characters) of typewritten text. In particular, the first chapter consists of 55438, the second-42615, the third-34120 characters.

CONTENT OF THE DISSERTATION PAPER

Abstract. The introduction analyzes the current state of the considered problem and shows the relevance of the selected topic.

The first chapter is devoted to the study and development of methods for solving the interval problem of partial Boolean programming. For this purpose, the concepts of optimistic, pessimistic, suboptimistic and subpessimistic solutions for interval partial Boolean knapsack problems and partial Boolean programming problems are introduced.

In paragraph 1.1 of the I chapter the following problem is considered:

$$\sum_{j=1}^{n}[\underline{c}_j,\overline{c}_j]x_j + \sum_{j=n+1}^{N}[\underline{c}_j,\overline{c}_j]x_j \to max \qquad (5)$$

$$\sum_{j=1}^{n}[\underline{a}_j,\overline{a}_j]x_j + \sum_{j=\mathrm{n}+1}^{N}[\underline{a}_j,\overline{a}_j]x_j \le [\underline{b},\overline{b}]\,, \qquad (6)$$

$$0 \le x_j \le 1, \left(j=\overline{1,N}\right), \qquad (7)$$

$$x_j = 1 \vee 0, \left(j=\overline{1,n}\right)\ (n \le N). \qquad (8)$$

Here, it is assumed $0<\underline{c}_j \le \overline{c}_j$, $0<\underline{a}_j \le \overline{a}_j$, $\left(j=\overline{1,N}\right)$, $0<\underline{b} \le \overline{b}$ and are integer.

To construct suboptimistic and subpessimistic solutions, the following definitions have been introduced:

Definition 1. An admissible solution $X^{op} = \left(x_1^{op}, x_2^{op}, \dots, x_N^{op}\right)$ of problem (5) - (8) is called optimistic if the inequality

$$\sum_{j=1}^{N}\underline{a}_j x_j^{op} \le b,$$

for $\forall b \in [\underline{b}, \overline{b}]$ and in this the value of the function

$$f^{op} = \sum_{j=1}^{N} \overline{c}_j x_j^{op}$$

will be the maximum.

Definition 2. An admissible solution $X^p = (x_1^p,\ x_2^p, \dots, x_N^p)$ of problem (5) - (8) is called pessimistic if the relation

$$\sum_{j=1}^{N} \overline{a}_j x_j^p \leq b,$$

for $\forall b \in [\underline{b}, \overline{b}]$ and the value of the function

$$f^p = \sum_{j=1}^{N} \underline{c}_j x_j^p$$

will be the maximum.

Definition 3. An admissible solution $X^{so} = (x_1^{so}, x_2^{so}, \dots, x_N^{so})$ of problem (5) - (8) is called suboptimistic if

$$\sum_{j=1}^{N} \underline{a}_j x_j^{so} \leq b,$$

for $\forall b \in [\underline{b}, \overline{b}]$ and the value of the function

$$f^{so} = \sum_{j=1}^{N} \overline{c}_j x_j^{so}$$

will take on a large number.

Definition 4. An admissible solution $X^{sp} = (x_1^{sp}, x_2^{sp}, \dots, x_N^{sp})$) of problem (5) - (8) is called subpessimistic if the relation

$$\sum_{j=1}^{N} \overline{a}_j x_j^{sp} \le b$$

for $\forall\, b \in [\underline{b}, \overline{b}]$ and for this value of the function

$$f^{sp} = \sum_{j=1}^{N} \underline{c}_j x_j^{sp}$$

will take on a large number.

Using the following criteria

$$j_* = arg \max_j \ (\overline{c}_j \,/\, \underline{a}_j) \tag{9}$$

$$j_* = arg \max_j \ (\underline{c}_j \,/\, \overline{a}_\mathrm{j}) \tag{10}$$

an algorithm for constructing a suboptimistic and subpessimistic solution to problem (5) - (8) has been developed.

It should be noted that formulas (9) and (10) can be taken as a criterion for choosing unknowns x_j for constructing suboptimistic or subpessimistic solutions, respectively. In this case, it is necessary to take into account the case in which intervals the found number j_* is included, i.e. $j_* \in [1, \dots, n]$ or $j_* \in [n+1, n+2, \dots, N]$.

Note that, similarly to the above, one can construct a subpessimistic solution $X^{sp} = \left(x_1^{sp}, x_2^{sp}, \dots, x_N^{sp}\right)$ to problem (5) - (8) using criterion (10).

The following theorems have been proved:

Theorem 1. Let the intervals $A = [a_1, a_2]$ и $B = [b_1, b_2]$ are given. If $0 < a_1 \le a_2$, $0 < b_1 \le b_2$, then $\min(a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2) = a_1 \cdot b_1$, $\max(a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2) = a_2 \cdot b_2$.

Theorem 2. Let the intervals $A = [a_1, a_2]$ и $B = [b_1, b_2]$. If $0 < a_1 \le a_2$, $0 < b_1 \le b_2$, then $\min(a_1/b_2, a_1/b_1, a_2/b_2, a_2/b_1) = a_1/b_2$, $\max(a_1/b_2, a_1/b_1, a_2/b_2, a_2/b_1) = a_2 / b_1$. Thus, $A \times B = [a_1 \cdot b_1,\ a_2 \cdot b_2]$, $A{:}B = [a_1/b_2, a_2/b_1]$.

The advantages of these theorems from those given in the book[4] is that the same result can be obtained proceeding from the above optimistic or pessimistic strategy by performing fewer operations.

Paragraph 1.2 considers the following problem with many restrictions:

$$\sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j + \sum_{j=n+1}^{N} [\underline{c}_j, \overline{c}_j] x_j \to max \qquad (11)$$

$$\sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j + \sum_{j=\mathrm{n}+1}^{N} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \le [\underline{b}_i, \overline{b}_i], \qquad (i = \overline{1,m}\,). \qquad (12)$$

$$0 \le x_j \le 1, \left(j = \overline{1,N}\right), \qquad (13)$$

$$x_j = 1 \vee 0, \left(j = \overline{1,n}\right), (n \le N). \qquad (14)$$

[4] Алефельд, Г. Введение в интервальные вычисления / Г. Алефельд, Ю. Херцбергер – Пер. с англ. Москва: Мир, – 1987. – 360 с.

Here, it is assumed that $0 < \underline{c}_j \leq \overline{c}_j$, $0 \leq \underline{a}_{ij} \leq \overline{a}_{ij}$, $0 < \underline{b}_i \leq \overline{b}_i$, $(i = \overline{1,m}\ ,\ j = \overline{1,N})$ are given integers. To construct suboptimistic and subpessimistic solutions of this problem, the concepts of optimistic, pessimistic, suboptimistic and subpessimistic solutions, similar to those in paragraph 1.1, are introduced[5].

The following criteria are derived for choosing the number j_* of unknowns x_j for constructing suboptimistic and subpessimistic solutions for problem (11) - (14).

$$j_* = arg \max_j \left(\overline{c}_j \ / \ \max_i \underline{a}_{ij} \right), \tag{15}$$

$$j_* = arg \max_j \left(\underline{c}_j \ / \ \max_i \overline{a}_{ij} \right). \tag{16}$$

Thus, criterion (15) can be used to construct a suboptimistic solution, and criterion (16) can be used for a subpessimistic solution. In this case, it is necessary to take into account the case in which the interval contains the found number j_*, i.e. $j_* \in [1, \dots, n] \equiv I$ or $j_* \in [n+1, n+2, \dots, N] \equiv R$.

Considering these circumstances, two methods have been developed for constructing an approximate solution.

In the first method, in the case when it is impossible to assign a unit to the unknown $x_{j_*}, (j_* \in R)$ for the first time, then for this unknown we take possible maximum fractional values, and the remaining variables will take on zero values .

In the second method, as soon as the unknown x_{j_*}, $(j_* \in R)$ cannot be assigned a unit for the first time, we fix the values of the unknowns found until then, for the remaining numbers j, $j \in I$ we

[5] Мамедов, К.Ш., Мамедова, А.Г. Понятия субоптимистического и субпессимистического решений и построения их в интервальной задаче Булевого программирования //−Украина: «Радиоэлектроника, Информатика, Управление», − 2016. № 3(38), − с. 99-108.

assign $x_j \coloneqq 0$ and for non-fixed unknowns x_j, $(j \in R)$ we construct a linear programming problem. Further, having solved the obtained problem, we add the found coordinates of the solutions to the previously fixed ones. The process of constructing suboptimistic and subpessimistic solutions begins from $X^{so} = (0,0,\dots,0)$, $X^{sp} = (0,0,\dots,0)$.

Paragraph 1.3 of the first chapter also considers problem (11) - (14), which is written in the following equivalent form for fixed b_i, $b_i \in [\underline{b}_i, \overline{b}_i]$, $(i = \overline{1,m})$:

$$\sum_{j=1}^{n}[\underline{c}_j, \overline{c}_j]x_j + \sum_{j=n+1}^{N}[\underline{c}_j, \overline{c}_j]x_j \rightarrow max, \tag{17}$$

$$\sum_{j=1}^{n}[\underline{\alpha}_{ij}, \overline{\alpha}_{ij}]x_j + \sum_{j=n+1}^{N}[\underline{\alpha}_{ij}, \overline{\alpha}_{ij}]x_j \leq 1, \ (i = \overline{1,m}), \tag{18}$$

$$0 \leq x_j \leq 1, \ \left(j = \overline{1,N}\right), \tag{19}$$

$$x_j = 1 \vee 0, \ \left(j = \overline{1,n}\right), \ (n \leq N). \tag{20}$$

Here $\underline{\alpha}_{ij} = \underline{a}_{ij}/b_i$, $\overline{\alpha}_{ij} = \overline{a}_{ij}/b_i$, $b_i \coloneqq 1$, $(i = \overline{1,m}; j = \overline{1,N})$. It's obvious that $0 \leq \underline{\alpha}_{ij} \leq 1$, $0 \leq \overline{\alpha}_{ij} \leq 1$, $(i = \overline{1,m}; j = \overline{1,N})$.

To construct a suboptimistic solution, the following criterion (21) is used:

$$j_* = arg \max_j \overline{Q}_j. \tag{21}$$

Here $\overline{Q}_j = \overline{c}_j / \underline{q}_j$, $\left(j = \overline{1,N}\right)$,

$$\underline{q}_j = \sum_{i=1}^{m} \underline{\alpha}_{ij}\, \underline{t}_i, \qquad \left(j = \overline{1,N}\,\right),$$

$$\underline{t}_i = 1 \,/\, (1 - \underline{r}_i), \;\; (i = \overline{1,m}), \tag{22}$$

$$\underline{r}_i = \sum_{j\in\omega} \underline{\alpha}_{ij}\,, \qquad \left(i = \overline{1,m}\right), \qquad \omega = \left\{j \middle| x_j^{so} = 1\right\}.$$

At the start of the process of constructing solutions, $\omega = \emptyset$ is taken. Similarly, you can build a subpessimistic solution.

As you can see, formula (22), which was called the penalty, occupies an important place here. Since this penalty increases non-linearly when constructing a solution, this method was called the nonlinearly increasing penalty method.

Paragraph 1.4 presents the results of experiments for various high-dimensional problems.

In the second chapter, methods for solving the interval problem of partial-integer programming are developed.

Paragraph 2.1 considers the following problem:

$$\sum_{j=1}^{n}\left[\underline{c}_j, \overline{c}_j\right] x_j + \sum_{j=n+1}^{N}\left[\underline{c}_j, \overline{c}_j\right] x_j \to \max, \tag{23}$$

$$\sum_{j=1}^{n}\left[\underline{a}_j, \overline{a}_j\right] x_j + \sum_{j=n+1}^{N}\left[\underline{a}_j, \overline{a}_j\right] x_j \le \left[\underline{b}, \overline{b}\right], \tag{24}$$

$$0 \le x_j \le d_j \quad \left(j = \overline{1,N}\right), \tag{25}$$

$$x_j, \text{ integer } \left(j = \overline{1,n}\right), (n \le N). \tag{26}$$

Here, it is assumed that $0 < \underline{c}_j \leq \overline{c}_j,\ 0 \leq \underline{a}_j \leq \overline{a}_j,$
$d_j > 0, \left(j = \overline{1, N}\right),\ \ 0 < \underline{b} \leq \overline{b}$ and are integers.

The concepts of optimistic, pessimistic, suboptimistic and subpessimistic solutions are introduced[6].

Based on some economic interpretation of problem (23) - (26), the following criteria were derived for choosing the number of unknowns for constructing suboptimistic and subpessimistic solutions, respectively:

$$j_* = arg\max_j(\overline{c}_j/\underline{a}_j), \qquad j_* = arg\max_j(\underline{c}_j/\overline{a}_j).$$

Here it is necessary to take into account the circumstances, in which sets the number j_*, is included, i.e. $j_* \in [1, \dots, n]$ or $j_* \in [n+1, n+2, \dots, N]$.

Based on the above criteria, algorithms have been developed for constructing suboptimistic and subpessimistic solutions of problem (23) - (26).

We take the notation $I = \{1, \dots, n\}$ and $R = \{n+1, n+2, \dots, N\}$. In addition, let us denote by S the set of numbers of unknowns to which nonzero values are assigned. Obviously, at the beginning S = ∅. To build a suboptimistic solution according to the criterion

$$j_* = arg\max_{j \in I \cup R}(\overline{c}_j/\underline{a}_j). \qquad (27)$$

2 cases are considered:
If $j_* \in I$, we accept

$$x_{j_*} = \min\left\{d_{j_*}, \left[(b - \sum_{i \in S}\underline{a}_i x_i)/\underline{a}_{j_*}\right]\right\}, \qquad S := S \cup \{j_*\},$$

[6] **Mammadli N.O.** An algorithm for the construction of suboptimistic and subpessimistic solutions of a mixed integer knapsack problem with interval data // ‒Baku: Transactions of Azerbaijan National Academy of Sciences Series of Physical-Technical and Mathematical Sciences, Informatics and Control Problems, ‒ 2019. vol. 39, No.6, Issue 2, №.6, ‒ p.74-80.

$I \coloneqq I \setminus \{j_*\}$. Here, the notation [z] means the integer part of z. Further, according to the formula (27), another number j_* is found.
If $j_* \in R$, we accept

$$x_{j_*} = \min\left\{d_{j_*}, (b - \sum_{i \in S} \underline{a}_i x_i)/\underline{a}_{j_*}\right\},$$

$S \coloneqq S \cup \{j_*\}, R \coloneqq R \setminus \{j_*\}$.

The following options should be considered here.

If

$$\min\left\{d_{j_*}, (b - \sum_{i \in S} \underline{a}_i x_i)/\underline{a}_{j_*}\right\} = d_{j_*},$$

then the computational process continues in the above manner according to criterion (27). If

$$\min\left\{d_{j_*}, (b - \sum_{i \in S} \underline{a}_i x_i)/\underline{a}_{j_*}\right\} = (b - \sum_{i \in S} \underline{a}_i x_i)/\underline{a}_{j_*},$$

Then we accept $x_j^{so} \coloneqq 0$ for all $j \notin S$ and the process of building a suboptimistic solution is completed.

The process ends and also when $I \cup R = \emptyset$.

Paragraph 2.2 considers the following problem:

$$\sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j]\, x_j + \sum_{j=n+1}^{N} [\underline{c}_j, \overline{c}_j]\, x_j \to \max, \tag{28}$$

$$\sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}]\, x_j + \sum_{j=n+1}^{N} [\underline{a}_{ij}, \overline{a}_{ij}]\, x_j \le [\underline{b}_i, \overline{b}_i],\ (i = \overline{1, m}), \tag{29}$$

$$0 \le x_j \le d_j, \ \left(j = \overline{1, N}\right), \tag{30}$$

$$x_j, \text{ integers}, \ \left(j = \overline{1, n}\right), \ (n \le N) \tag{31}$$

Here the coefficients of problem (28) - (31) are integer and satisfy the following conditions: $0 < \underline{c}_j \le \overline{c}_j, 0 \le \underline{a}_{ij} \le \overline{a}_{ij},$
$0 < \underline{b}_i \le \overline{b}_i, \ d_j > 0, \left(i = \overline{1, m}; \ j = \overline{1, N}\right).$

For this problem, using the analogous concepts of optimistic and pessimistic solutions, 2 methods for constructing an approximate solution have been developed, which we call suboptimistic and subpessimistic.

For this, the following criteria for choosing unknown j_* are derived

$$j_* = arg\left(\max_j\left(\overline{c}_j / \max_i \underline{a}_{ij}\right)\right) \tag{32}$$

$$\boldsymbol{j_* = arg\left(max_j\left(\underline{c}_j \,/\, max_i\, \overline{a}_{ij}\right)\right)} \tag{33}$$

appropriate for building suboptimistic and subpessimistic solutions.

Paragraph 2.3 also considers problem (28) - (31). For this problem, similar definitions 1-4 are introduced, which are more general. In order to construct a suboptimistic and subpessimistic solution in problem (28) - (31) we fix some $b_i \in \left[\underline{b}_i, \overline{b}_i\right]$, $\left(i = \overline{1, m}\right)$ and divide both sides of inequality (29) by $b_i \in \left[\underline{b}_i, \overline{b}_i\right]$, $\left(i = \overline{1, m}\right)$. As a result, we get the following equivalent problem:

$$\sum_{j=1}^{n}\left[\underline{c}_j, \overline{c}_j\right] x_j + \sum_{j=n+1}^{N}\left[\underline{c}_j, \overline{c}_j\right] x_j \to \max, \tag{34}$$

$$\sum_{j=1}^{n}[\underline{\alpha}_{ij},\overline{\alpha}_{ij}]\,x_j + \sum_{j=n+1}^{N}[\underline{\alpha}_{ij},\overline{\alpha}_{ij}]\,x_j \le 1\,,\ (i=\overline{1,m}), \quad (35)$$

$$0 \le x_j \le d_j,\ \left(j=\overline{1,N}\right), \quad (36)$$

$$x_j, \text{ целые}, \left(j=\overline{1,n}\right), (n \le N). \quad (37)$$

Here $\underline{\alpha}_{ij} = \underline{a}_{ij}/b_i,\ \overline{\alpha}_{ij} = \overline{a}_{ij}/b_i,\ 0 \le \underline{\alpha}_{ij} \le 1, 0 \le \overline{\alpha}_{ij} \le 1,$ $(i=\overline{1,m}), \left(j=\overline{1,N}\right)$. Further, for different number j_*, a positive value for $x_{j_*}^{so}$ is sequentially assigned. Finding the number j_* is based on the following definition:

Definition 5. The number $\underline{t}_i = 1/(1-\underline{r}_i),\ (i=\overline{1,m})$ is called the penalty for using the remaining right-hand sides of system (35) to accept the further positive value for $x_j^{so}, \left(j=\overline{1,N}\right)$, where

$$\underline{r}_i = \sum_{j\in\omega}\underline{\alpha}_{ij}\,x_j^{so}, \qquad (i=\overline{1,m}), \qquad \omega = \{j, \left|x_j^{so} > 0\right.\}.$$

To construct a suboptimistic solution, the following criterion for choosing the number номера j_* is used:

$$j_* = arg\max_{j\in I\cup R}\{\,\overline{c}_j/\underline{q}_j\}$$

$$\underline{q}_j = \sum_{j=1}^{N}\underline{\alpha}_{ij}\,\underline{t}_i, \qquad \left(j=\overline{1,N}\right),$$

Chapter III is devoted for assessing the error of suboptimistic and subpessimistic decisions from optimistic and pessimistic ones. For this purpose, the upper bound for the maximum value of the objective function is determined.

Paragraph 3.1 considers the interval partial Boolean knapsack problem (9) - (12).

For this problem, the following functions are built:

$$L^{op}(\lambda) = \sum_{j\in\omega_1^{op}} \overline{c}_j + \sum_{j\in\omega_2^{op}} \overline{c}_j + \left(b - \sum_{j\in\omega_1^{op}} \underline{a}_j - \sum_{j\in\omega_2^{op}} \underline{a}_j\right)\cdot\lambda,$$

$$L^{p}(\lambda) = \sum_{j\in\omega_1^{p}} \underline{c}_j + \sum_{j\in\omega_2^{p}} \underline{c}_j + \left(b - \sum_{j\in\omega_1^{p}} \overline{a}_j - \sum_{j\in\omega_2^{o}} \overline{a}_j\right)\cdot\lambda.$$

Here $b \in \left[\underline{b}, \overline{b}\right]$ is fixed and

$$\omega_1^{op} = \left\{1 \le j \le n \middle| \overline{c}_j - \underline{a}_j \cdot \lambda > 0\right\},$$

$$\omega_2^{op} = \left\{n+1 \le j \le N \middle| \overline{c}_j - \underline{a}_j \cdot \lambda > 0\right\},$$

$$\omega_1^{p} = \left\{1 \le j \le n \middle| \underline{c}_j - \overline{a}_j \cdot \lambda > 0\right\},$$

$$\omega_2^{p} = \left\{n+1 \le j \le N \middle| \underline{c}_j - \overline{a}_j \cdot \lambda > 0\right\}.$$

The following theorems are proved:

Theorem 1. The following inequalities are valid:

$$f_*^{op} \le \min_{\lambda\ge 0} L^{op}(\lambda), \qquad f_*^{p} \le \min_{\lambda\ge 0} L^{p}(\lambda),$$

where f_*^{op} and f_*^{p} are the optimistic and pessimistic values of the objective function, respectively.

Theorem 2. The functions $L^{op}(\lambda)$ and $L^{p}(\lambda)$ are continuous, piecewise linear, non-differentiable, and convex.

Theorem 3. The minimum values of $L^{op}(\lambda)$ and $L^{p}(\lambda)$ coincide with the maximum values of the objective functions of the optimistic and pessimistic continuous problem (9) - (12), respectively. Algorithms for minimizing these functions have been developed.

In paragraph 3.2, the following majorizing functions are constructed with respect to the objective function for the interval partial Boolean programming problem (15) - (18).

$$L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m) = \sum_{j\in\omega_1^{op}} \overline{c}_j + \sum_{j\in\omega_2^{op}} \overline{c}_j +$$

$$+\sum_{i=1}^{m}\left(b_i - \sum_{j\in\omega_1^{op}} \underline{a}_{ij} - \sum_{j\in\omega_2^{op}} \underline{a}_{ij}\right)\lambda_i,$$

$$L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m) = \sum_{j\in\omega_1^{p}} \underline{c}_j + \sum_{j\in\omega_2^{p}} \underline{c}_j +$$

$$+\sum_{i=1}^{m}\left(b_i - \sum_{j\in\omega_1^{p}} \overline{a}_{ij} - \sum_{j\in\omega_2^{p}} \overline{a}_{ij}\right)\lambda_i,$$

Here

$$\left.\begin{aligned} \omega_1^{op} &= \left\{1 \le j \le n \middle| \overline{c}_j - \sum_{i=1}^{m} \underline{a}_{ij}\lambda_i > 0\right\}, \\ \omega_2^{op} &= \left\{n+1 \le j \le N \middle| \overline{c}_j - \sum_{i=1}^{m} \underline{a}_{ij}\lambda_i > 0\right\} \end{aligned}\right\}$$

$$\left.\begin{aligned} \omega_1^{p} &= \left\{1 \le j \le n \middle| \underline{c}_j - \sum_{i=1}^{m} \overline{a}_{ij}\lambda_i > 0\right\}, \\ \omega_2^{p} &= \left\{n+1 \le j \le N \middle| \underline{c}_j - \sum_{i=1}^{m} \overline{a}_{ij}\lambda_i > 0\right\} \end{aligned}\right\}.$$

It is shown that the functions $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m)$ are functions of Lagrange type. The following theorems have been proved:

Theorem 4. For the optimistic value f_*^{op} and the pessimistic value f_*^{p} of the objective function of problem (15) - (18), the following inequalities are valid:

$$f_*^{op} \le \min_{\lambda_i \ge 0} L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m), \qquad f_*^{p} \le \min_{\lambda_i \ge 0} L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m).$$

Theorem 4 shows that by minimizing the functions $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ or $L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m)$, one can find upper bounds for f_*^{op} or f_*^{p}, respectively. For this, some properties of the function $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m)$ have been proved, which create a mathematical basis for minimization.

Theorem 5. The functions $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m)$ are piecewise linear, continuous, non-differentiable, and convex.

Corollary. Theorem 5 immediately implies that the functions $L^{op}(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $L^{p}(\lambda_1, \lambda_2, \dots, \lambda_m)$ have unique minima. Consequently, the development of algorithms for minimizing these functions takes place.

In paragraph 3.3, algorithms for minimizing these functions of the steepest coordinate descent type are developed.

Finally, **paragraph 3.4** presents the results of numerous computational experiments for high-dimensional problems with random coefficients.

The appendix provides a program for finding suboptimistic and subpessimistic solutions of the interval partial-integer knapsack problem, with an estimate of the deviation from the optimistic and pessimistic values of the objective function, respectively.

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THE MAIN RESULTS OF THE DISSERTATION

1. For an interval partial Boolean programming problem with one and many constraints, the concepts of optimistic, pessimistic, suboptimistic and subpessimistic solutions are introduced. Methods for finding suboptimistic and subpessimistic solutions to these problems have been developed.
2. Based on the economic interpretation of the interval problem of partial-Boolean programming, the concept of a nonlinearly increasing penalty is introduced. On the basis of this, a nonlinearly increasing penalty method is proposed for constructing suboptimistic and subpessimistic solutions to this problem.
3. More general concepts of admissible, optimistic, pessimistic, suboptimistic and subpessimistic solutions are introduced for an interval partial integer programming problem with one and many constraints. Further, methods for constructing suboptimistic and subpessimistic solutions to these problems are proposed.
4. Using the concept of nonlinearly increasing penalty introduced in the thesis for partially integer programming problems with interval coefficients, a method for finding suboptimistic and subpessimistic solutions has been developed.
5. In order to determine the proximity of suboptimistic and subpessimistic solutions to optimistic and pessimistic ones, an algorithm for finding the upper bound of the optimistic and pessimistic values of the functional for an interval partial Boolean programming problem with one and many constraints has been developed.

6. The algorithms of all the proposed methods in the dissertation work have been developed and the programs of these algorithms have been compiled. Using these programs, numerous computational experiments were carried out on various random problems of high dimension.
7. The dissertation work includes a program for finding suboptimistic and subpessimistic solutions to the interval partial-integer knapsack problem, as an appendix.

The main results of dissertation are published in the following academic papers of author:

1. Мамедов К.Ш., Мамедли Н.О. Построение субоптимистического и субпессимистического решений интервальной частично-Булевой задачи о ранце // Əmir Şamil oğlu Həbibzadənin anadan olmasının 100-cü ildönümünə həsr olunmuş "Funksional Analiz və Onun Tətbiqləri" adlı konfransının Materialları,− Bakı: − 2016, − s.160-161.
2. Мамедов К.Ш., Мамедли Н.О. Построение субоптимистического и субпессимистического решений в интервальной задаче частично-Булевого программиро-вания // "Riyaziyyatın Tətbiqi Məsələləri və Yeni İnfor-masiya Texnologiyları" adlı III Respublika Elmi Konfrans Matererialları, − Bakı: − 15-16 dekabr, − 2016, − s.113-114.
3. Мамедов К.Ш., Мамедли Н.О. Методы построения субоптимистического и субпессимистического решений частично-Булевой задачи о ранце с интервальными данными //− Bakı: AMEA-nın xəbərləri:fizika-texnika və riyaziyyat elmləri seriyası, − 2016. vol.XXXVI, №6, − s.6-13.
4. Мамедов К.Ш., Мамедли Н.О. Решение частично-целочисленной задачи о ранце с интервальными данными //

Riyaziyyatın Nəzəri və Tətbiqi Problemləri Beynəlxalq Elmi Konfransın Materialları, − Bakı: − 25-26 may, − 2017, − s.229-230.
5. Mammadov K.Sh., Mammadli N.O. Two methods for construction of suboptimistic and subpessimistic solutions of the interval problem of mixed-Boolean programming // − Ukraine: Journal “Radio Electronics, Computer Science, Control”, − 2018. №3(46), − p.57-67.
6. Мамедов К.Ш., Мамедли Н.О. Методы приближённого решения задач частично-Булевого программирования с интервальными данными //− Bakı: AMEA-nın xəbərləri:fizika-texnika və riyaziyyat elmləri seriyası, − 2018. vol.XXXVIII, №3, − s. 27-35.
7. Mammadov K. Sh., Mammadli N.O. Method of finding sub-optimistic and subpessimistic solutions of the mixed-Boolean programming problem with interval data // Materials of “The 6-th Int. Conf. on Control and Optimization with Industrial Applications”, −Baku: − 11-13 July,− 2018. vol.II, −p.217-219.
8. Mammadov K. Sh., Mammadli N.O. Methods for finding suboptimistic and subpessimistic solutions to interval part of the integer programming problem // −Baku: Transactions of Azerbaijan National Academy of Sciences Series of Physical-Technical and Mathematical Sciences, Informatics and Control Problems, − 2019. vol.39, №.3, − p.56-63.
9. Mammadli N.O. An algorithm for the construction of suboptimistic and subpessimistic solutions of a mixed integer knapsack problem with interval data // − Baku: Transactions of Azerbaijan National Academy of Sciences Series of Physical-Technical and Mathematical Sciences, Informatics and Control Problems, − 2019. vol. 39, No.6, Issue 2, №.6, − p.74-80.

10. **Mammadov K.Sh., Mammadli** N.O. Approximate solutions of the interval problem of mixed-integer programming // − Bakı: AMEA “Məruzələr”, − 2019. №1, − p.25-28.
11. **Мамедов К.Ш., Мамедли Н.О.** Построение субоптимистического и субпессимистического решений частично-целочисленной интервальной задачи о ранце // −Воронеж: Актуальные направления научных исследо-ваний XXI века: “Теория и практика”, − 2019.vol.7, №1(44), − с. 247-251.
12. **Мамедов К.Ш., Мамедли Н.О.** Метод построения приближённого решения интервальной задачи частично-целочисленого программирования //− Москва: Евразий-ский Союз Ученых (ЕСУ), −2019. №4(61), − с. 29-36.
13. **Mammadli, N.O.** The Determination of an Upper Bound of the Maximum Value of the Objective Function in an Interval Mixed Boolean Knapsack Problem //− Москва: Евразийский Союз Ученых (ЕСУ),− 2019. № 9 (66), − p.49-54.
14. **Мамедов К.Ш., Мамедли Н.О.** Приближённые решения интервальной задачи частично-целочисленного программирования // Riyaziyyatin Fundamental Problemləri və Intellektual Texnologiyalarin Təhsildə Tətbiqi Respublika Elmi Konfransinin Materiallari, −Sumqayıt: 3-4 iyul, 2020, − c.71-75.
15. **Mammadov K .Sh., Mammadli N.O.** A method for finding the upper bound of optimistic and pessimistic values of the functional in the interval Boolean programming problem // Materials of “The 7-th Int. Conf. on Control and Optimization with Industrial Applications”, −Baku: 26-28 august, − 2020. vol. I, p.251-253.
16. **Мамедов К.Ш., Мамедли Н.О.** Построение функции типа Лагранжа в интервальной задаче частично-Булевого программирования // Москва: Евразийский Союз Ученых (ЕСУ), − 2020. Том 6, №9(78),− с. 46-52.

Personal contribution of the author:

in works [1-3] building a model and conducting computational experiments;

[4]–development algorithms and compilation computational experiments;

[5]–the development of one of two methods, compilation of all algorithms and computational experiment;

[6-8]– drawing up models, expanding some concepts for a more general problem and participating in the development of solution methods;

[10-12] – the development of algorithms for the proposed methods;

[14,15,16]– compilation of an algorithm for minimization of a majorizing function of the Lagrange type.

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