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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

ON GEOMETRY OF SPECIAL RIEMANNIAN METRICS IN DIFFERENTIABLE MANIFOLDS AND THEIR BUNDLES

Specialty:	1204.01 – Geometry
Field of science:	Mathematics
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Baku – 2021

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«23» __ 2021 il

GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic. One of the most developed fields of modern differential geometry is the theory bundle spaces. The problem related to investigations of differential-geometrical structures, including vector and tensor fields, affine connections, Riemannian metrics on smooth manifiolds and in their bundle spaces is one of the most actual issues. Differential-geometrical structures determined as lifts (horizontal, complete and vertical lifts) of similar differential-geometrical structures are of more interest. This in the first place is connected with the fact that these lifts repeat the main properties of structures given in the base. Vertical, complete and horizontal lifts of vector fields to tangent bundle space were structured by S.Sasaki, S.I.Shihara, K.Yano, Sh. Kobayashi. In the case when the base manifold was a Riemannian manifiold, S.Sasaki has determined a special type Riemannian metric in the tangent bundle space and afterwards this metric was called Sasaki metric or a diagonal lift of Riemannian metric. Construction of lifts enabled to study almost complex and paracomplex (or almost product) structures in a tangent bundle space. The existence of a dual structure in tangent bundle space allowed A.R.Shirokov to interprete this bundle space as a manifold structured over the algebra of dual members. This makes construction of lifts of tensor fields and affine connections in tangent bundle spaces more easier. This idea was developed when studying semi-tangent bundle spaces. When studying co-tangent bundle spaces and bundle spaces of linear frames the results similar to the results obtained when researching tangent bundle spaces were obtained. Sasaki metric in co-tangent bundle space was constructed by Mok and its properties were sutdied by different scientists. Various modifications of Sasaki metric in bundle of frames were mainly studied by O.Kovalski and M.Sekizava. It should be noted that the Sasaki metric and also Cheeger-Cromoll metric are from these classes. Some interesting results on the study of differential-geometrical structures, also tensor fields, affine connection lifts and various metrics in different type tensor bundle spaces on smooth manifolds were obtained. In spite of the fact that the issue of construction of different new types of natural Riemannian metrics in tangent and co-tangent bundle spaces, study of Levi-Chivita connection are the problems distinguished by their actuality, this problem has not been investigated sufficiently.

The topic of the represented dissertation work is related to the solution of this problem. In this sense the topic of the dissertation work is actual.

Object and subject of the study. Geodesic curves of natural Riemannian metrics in tangent and cotangent bundle spaces, their properties and study of the Levi-Chivita connection.

Goal and objectives of the study. The goal of the study is to construct new types of the class of natural Riemannian metrics being a wide class of Riemannian metrics in tangential and cotangent bundles.

Research methods. In the research of the problems under consideration the methods of tensor calculus on smooth manifolds were used.

The main points of the study. The following main scientific results were obtained:

1. Interpretations of geodesic curves of Cheeger-Gromoll metrics in tangent bundles;
2. Transformation of the complete lift of a symplectic metric in a tangent bundle space into a natural symplectic metrics of the cotangent bundle space in canonical symplectic mapping;
3. Necessary and sufficient conditions for the images of complete lifts of vector, affinor and (1,2) type tensor fields to be again complete lifts in a canonical symplectic mapping;
4. The conditions for the vector fields with respect to the Peterson extended Riemannian metrics in cotangent bundle to be a Killing vector and to be Norden metrics.

Scientific novelty of the study. The main results obtained in the study are new and there are the followings:

1. Interpretation of geodesic curves of Gheeger-Gromoll metrics was given in tangent bundle spaces;
2. It was shown that complete lift of a simplectic metrics in the tangent bundle space is transformed into a natural symplectic

metrics of cotangent bundlespace in canonical symplectic mapping;

3. Necessary and sufficient conditions for the images of complete lifts of vector, affine and (1,2) type tensor fields in canonical symplectic mapping to be again complete lifts;
4. The conditions for vector fields with respect to extended Riemannian metrics in the Peterson's sense in cotangent bundles to be a Killing vector and the conditions for them to be a Norden metric.

Theoretical and practical value of the study. The main results obtained in the dissertation work are mainly of theoretical character. The results obtained in the dissertation work and the used methods can be used in teaching of specialty courses.

Approbation and application. The results of the dissertation were reported at the Scientific conference devoted to honoured scientist, acad. A.I.Huseynov's 100-th jubilee (Baku, 2007), at the International scientific conference "Mathematical theories, problems in their applications and teaching" held at Ganja State University (Ganja, 2008), at the Republican scientific conference of Students, masters and young researchers devoted to 85 years of the National leader Haydar Aliyev (Baku, 2008), at the Republican scientific conference "Actual problems of Mathematics and Mechanics" devoted to 90 years of the National leader Haydar Aliyev (Baku, 2014), at the scientific conference "Actuals problems of Mathematics and Mechanics" devoted to 95 years of Baku State University (Baku, 2014), at the International scientific conference devoted to 85-th jubilee of the honored scientist, former rector of BSU, corr.-member of ANAS, prof. Y.C.Mammadov (Baku, 2015), at the Republican scientific conference "Actual problems of Mathematics and Mechanics" devoted to 95 years of the National leader Haydar Aliyev (Baku, 2015), at the International conference on "Modern problems of geometry and topology and their applications" (Uzbekistan, 2019).

The organization where the work was executed. Dissertation work was executed in the chair of "Algebra and Geometry" of "Mechanics-Mathematics" department of Baku State University.

Authors personal contribution. The results obtained in the dissertation belong to the applicant.

Published scientific works. The main results of the dissertation work were published in applicants 7 scientific works, including 2 single-authored, 3 in scientific editions incuded in “Web of Science” database. Furthermore, the results obtained in the dissertation were reported at international level 3 and republican level 5 scientific conferences one of them was published abroad.

Total volume of the dissertation work indicating separate structural units of the work in signs. The dissertation work consists of introduction, three chapters, conclusions (title page–424 signs, table of contents – 2875 signs, introduction – 24000 signs, chapter I - 40000 signs, chapter II–66000 signs, chapter III -76000 signs, conclusions–602 signs) and list of references consisting of 106 names. The total volume of the dissertation work is 209901 signs.

CONTENT OF THE DISSERTATION WORK

Let us give brief review of the dissertation work consisting of 3 chapters.

In the introduction we give brief review of works related to the dissertation work, substantiate actuality of the dissertation work, give main results obtained in the work and compare them with the results of another works.

Chapter I consists of three subchapters. Main notions on Riemannian metrics on differentiable manifolds, affine structures and bundle spaces are given in chapter 1.

Definition of the notion of a chart, atlas, differentiable manifold, affine connection, curvature and torsion tensors on it is given in subchapter 1 of chapter 1.

In subchapter 2 of chapter 1 we give definition of S structure on a differentiable manifold M_n from the class C^{∞}. Then we give definition to a polyaffinor structure or simply a Π – structure.

At the end of this subchapter we give definition of Norden metric and Norden manifold (M_{2n},φ,g).

If (M_{2n},φ,g) is a Norden manifold possesing g holomorphic Norden metric, then it is said that (M_{2n},φ,g) is a holomorphic Norden manifold.

In subchapter 3 of chapter 1 we give definition of bundle spaces of a differentiable manifold. Some examples on vector bundles are also given.

In chapter II natural metrics in tangent bundle spaces are considered, their geometrical structures are constructed.

In subchapter 1 of chapter 2 a wide information on Sasaki metrics in a tangent bundle on Riemannian manifold is given. The Levi-Chivita connection of tangent bundle with respect to $\hat{g}$ Sasaki metrics is determined.

Formulas for calculating curvature tensor of Levi-Chivita connections of Sasaki metric in a tangent bundle and also Riemannian curvature tensor of tangent bundle $(T(M_n),\hat{g})$ with Sasaki metric are given.

In subchapter 2 of chapter 2 definition of the Cheeger-Gromoll metric of the tangent bundle $T(M_n)$ is given and the Levi-Chivita connection of the tangent bundle possesing this metric is determined.

Taking into account that the Levi-Chivita connection was determined, we calculate the Riemannian curvature tensor of the tangent bundle $T(M_n)$.

In subchapter 3 of chapter 2 we study some features of Cheeger-Cromoll metric in the adapted frames in tangent bundle on Riemannian manifold. The Cheeger-Gromoll metric ${}^{CG}g$ on tangent bundle $T(M_n)$ is determined for all vector fields $X,Y\in\Im_0^1(M_n)$ as follows:

$$ {}^{CG}g\left({}^{H}X,{}^{H}Y\right)={}^{V}\left(g(X,Y)\right), \qquad (1)$$

$$ {}^{CG}g\left({}^{H}X,{}^{V}Y\right)=0, \qquad (2)$$

$$ {}^{CG}g\left({}^{V}X,{}^{V}Y\right)=\frac{1}{1+r^{2}}\left[{}^{V}\left(g(X,Y)\right)+\left(\gamma g_{X}\right)+\left(\gamma g_{Y}\right)\right], \tag{3} $$

where

$$ {}^{V}\left(g(X,Y)\right)=\left(g(X,Y)\right)\circ\pi. $$

It is clear that the ${}^{CG}g$ Cheeger-Cromoll metric belongs to the class of natural metrics. (It should be noted that when we say a natural metric we mean a metrics determined by the conditions (1) and (2)).

In subchapter 4 of chapter 2 we give a theorem on Levi-Chivita connection of Cheeger-Cromoll metric in the adapted frame:

Theorem 1. Let (M_n, g) be a Riemannian manifold and $T(M_n)$ its tangent bundle space with ${}^{CG}g$ Cheeger-Gromoll metric. Then the apropriate ${}^{CG}\nabla$ Levi-Chivita connection satisfies the following conditions for $\forall X,Y\in\Im_{0}^{1}(M_n)$:

$$ \begin{cases} {}^{CG}\nabla_{{}^{H}X}{}^{H}Y={}^{H}\left(\nabla_{X}Y\right)-\frac{1}{2}{}^{V}\left(R(X,Y)y\right), \\ {}^{CG}\nabla_{{}^{H}X}{}^{V}Y=\frac{1}{2\alpha}{}^{H}\left(R(y,Y)X\right)+{}^{V}\left(\nabla_{X}Y\right), \\ {}^{CG}\nabla_{{}^{V}X}{}^{H}Y=\frac{1}{2\alpha}{}^{H}\left(R(y,X)Y\right), \\ {}^{CG}\nabla_{{}^{V}X}{}^{V}Y=-\frac{1}{\alpha}\left({}^{CG}g\left({}^{V}X,\gamma\delta\right){}^{V}Y+{}^{CG}g\left({}^{V}Y,\gamma\delta\right){}^{V}X\right)+ \\ +\frac{1+\alpha}{\alpha}\,{}^{CG}g\left({}^{V}X,{}^{V}Y\right)\gamma\delta-\frac{1}{\alpha}\,{}^{CG}g\left({}^{V}X,\gamma\delta\right)\,{}^{CG}g\left({}^{V}Y,\gamma\delta\right)\gamma\delta, \end{cases} \tag{4} $$

where R and $\gamma\partial$, are respectively the curvature tensor of the connection ∇ and a canonical vertical field with the components

$$ \gamma\delta=\begin{pmatrix}0\\ x^{\bar{i}}\delta_{\bar{i}}^{\bar{j}}\end{pmatrix}=\begin{pmatrix}0\\ x^{\bar{j}}\end{pmatrix}=x^{\bar{j}}\partial_{\bar{j}}=x^{\bar{j}}e_{(\bar{j})}, $$

on $T(M_n)$.

With respect to the adapted frame $\{e_\alpha\}$ in the tangent bundle $T(M_n)$ we write the expression

$$^{CG}\nabla_{e_\alpha} e_\beta = {}^{CG}\Gamma^{\gamma}_{\alpha\beta} e_\gamma \, ,$$

where $^{CG}\Gamma^{\gamma}_{\alpha\beta}$ are the Christoffel symbols for ^{CG}g Cheeger-Cromoll metrics.

In subchapter 5 of chapter 2 we study geodesic curves of Cheeger-Cromoll metric in tangent bundle.

Theorem 2. Let $\tilde{c}$ be a curve on a tangent bundle $T(M_n)$ and locally expressed by $x^h = x^h(t), x^{\bar{h}} = y^h(t)$ with respect to the induced coordinates $\left(x^h, x^{\bar{h}}\right)$. Then the curve $\tilde{c}$ is a geodesic curve of the metric ^{CG}g if the following equations are satisfies:

$$\begin{cases} (a)\ \dfrac{\delta^2 x^h}{dt^2} + \dfrac{1}{\alpha} R^h_{kji} y^k \dfrac{\delta y^j}{dt} \dfrac{dx^i}{dt} = 0, \\ (b)\ \dfrac{\delta^2 y^h}{dt^2} + \left[-\dfrac{1}{\alpha}\left(y_j \delta^h_i + y_i \delta^h_j \right) + \dfrac{1+\alpha}{\alpha} g_{ij} y^h - \dfrac{1}{\alpha} y_j y_i y^h \right] \dfrac{\delta y^j}{dt} \dfrac{\delta y^i}{dt} = 0, \end{cases}$$

where $y^i = x^{\bar{i}}$.

Assume that $C = \pi \circ C^H$ is a geodesic curve of the connection ∇ on M_n. Then $\dfrac{\delta^2 x^h}{dt^2} = 0$.

Taking into account this condition and the condition $\dfrac{\delta y^j}{dt} = \dfrac{\delta X^h}{dt} = 0$ we get the following conclusion.

Theorem 3. The horizontal lif of the geodesic line on M_n is always geodesic curve in the tangnt bundle $T(M_n)$ with the metric ^{CG}g.

Now let us assume that $C = \pi \circ C^*$ is a geodesic curve of the connection ∇ on M_n, i.e.

$$\frac{\delta^2 x^h}{dt^2} = \frac{\delta}{dt}\left(\frac{dx^h}{dt}\right) = 0.$$

On the other hand, from the definition of the horizontal lift of the curve we have

$$\frac{\delta^2 y^h}{dt^2} = \frac{\delta}{dt}\left(\frac{dx^h}{dt}\right) = 0. \tag{5}$$

Then in equations (3) and (5) we can easily see that a natural lift of the curve determined by the equations $x^h = x^h(t)$ on M_n is a geodesic in the tangent bundle $T(M_n)$ with Cheeger-Gromoll metric ${}^{CG}g$.

In subchapter 6 of chapter 2 we study a deformed complete lift of Riemannian metric in a tangent bundle.

The notion related to lift is one of the important notions of differential geometry. The study of differential-geometrical objects on a tangent bundle and dual holomorphic objects on a dual-holomorphic manifold $X_n(R(\varepsilon))$ enables to determine a new class (deformed complete lifts) of lifts in a tangent bundle.

At first, deformed complete lifts of functions in a bundle are studied:

So, the following theorem is valid:

Theorem 4. Assume that g is a Riemannian metric on M_n, h is any tensor field of type $(0,2)$. Then in this case the tensor ${}^{Def}g = {}^{C}g + {}^{V}h$ is a Riemannian metric on a tangent bundle $T(M_n)$.

In subchapter 7 of chapter 2 we consider some problems related to lifts in symplectic geometry. In Riemannian geometry a canonical isomorphism is an isomorphism constructed between tangent and cotangent bundle of the Riemannian manifold by means metric. There are also similar isomorphisms among simplectic manifolds.

In the base manifold, the theory of continuation (lifts) of tensor fields to tangent and cotangent bundles was developed by Yano and

Ishihara. In this subchapter the main goal is to study transformation of lifts by means of symplectic canonical isomorphisms.

Theorem 5. Assume that (M,ω) is a symplectic manifold, ${}^{C}X_{T}$ and ${}^{C}X_{T^*}$ are complete lifts to tangent bundle $T(M)$ and cotangent bundle $T^*(M)$ respectively. If X is a symplectic vector field, then ${}^{C}X_{T}$ and ${}^{C}X_{T^*}$ is ω^b-related, i.e. $(\omega^b)_* {}^{C}X_T = {}^{C}X_{T^*}$.

Diffeomorphism of two symplectic manifolds $f:(M,\omega)\to(N,\omega')$ is called a symplectic morphism if $f^*\omega'=\omega$, here $f^*-$is a pullback of the diffeomorphism f.

Theorem 6. Assume that ω is a pure symplectic 2-form with respect to the $(1,2)$ type skew symmetric tensor field S on a symplectic manifold (M,ω) and assume that ${}^{C}S_{TM}$ and ${}^{C}S_{T^*M}$ are complete lifts of tangent and cotangent bundles, respectively. If ω symplectic 2-form satisfies the Yano-Ako equation

$$S_{ji}^{m}\partial_m\omega_{hs}-\left(\partial_j S_{hi}^{m}\right)\omega_{ms}-S_{hi}^{m}\partial_j\omega_{ms}-\left(\partial_i S_{jh}^{m}\right)\omega_{ms}-$$
$$-S_{jh}^{m}\partial_i\omega_{ms}+\omega_{ms}\partial_h S_{ji}^{m}+\omega_{hm}\partial_s S_{ji}^{m}=0$$

then the complete lift ${}^{C}S_{T^*M}$ is the transformation of the complete lift ${}^{C}S_{TM}$ as a result of canonical isomorphism

$$\omega^{\#}:T^*(M)\to T(M).$$

Chapter 3 considers metrics in cotangent bundle spaces.

In subchapter 1 of chapter 3 we consider Riemannian extension in Peterson's sense in tangent bundle spaces, construct the expression of Levi-Chivita connection in adapted frame.

With respect to the adapted frame $\{\tilde{e}_\beta\}$ of the covariant derivatives ${}^{C}\nabla\,{}^{H}X$ and ${}^{C}\nabla^{V}\omega$ we have

$$\left({}^{C}\nabla_{\gamma}\,{}^{H}\tilde{X}^{\alpha}\right)=\begin{pmatrix}\nabla_k X^i & 0\\ \frac{1}{2}p_a\left(R_{kji}{}^{a}-R_{jik}{}^{a}+R_{ikj}{}^{a}\right)X^i & 0\end{pmatrix} \tag{6}$$

and

$$\left({}^{C}\nabla_{\gamma}\,{}^{v}\tilde{\omega}^{\alpha}\right)=\begin{pmatrix}0 & 0\\ \nabla_k\omega_i & 0\end{pmatrix} \tag{7}$$

We prove that

$$\left({}^{C}\nabla_{\gamma}\,{}^{C}\tilde{X}^{\alpha}\right)=\begin{pmatrix}\nabla_k X^i & 0\\ -p_h\nabla_k\nabla_i X^h+\frac{1}{2}p_a\left(R_{kji}{}^{a}-R_{jik}{}^{a}+R_{ikj}{}^{a}\right)X^j-\nabla_i X^k & \end{pmatrix}. \tag{8}$$

From relation (7) we get the validity of the following theorem.

Theorem 7. The vertical lift of a covector field ω to $T^*(M_n)$ with metric ${}^{R}\nabla$ is parallel if and only if a given covector field ω is parallel with respect to ∇ .

İf the manifold M_n has g pseudo-Riemannian metric g, then according to the equality

$$\begin{aligned}p_a R_{kji}{}^{a} X^j &= p_a X^j\left(R_{kjis}g^{sa}\right)=p_a X^j\left(R_{iskj}g^{sa}\right)=\\ &=p_a X^j\left(-R_{isjk}g^{sa}\right)=p_a X^j\left(-R_{isj}{}^{t}g_{tk}g^{sa}\right)=-p_a g_{tk}g^{sa}\nabla_{[i}\nabla_{s]}X^t\end{aligned} \tag{9}$$

from relation (6) and (8) we get the following theorem:

Theorem 8. If M_n has a pseudo-Riemannian metric g and the Levi-Civita connection ∇ of g and $T^*(M_n)$ has the Riemannian extension ${}^{R}\nabla$ as its metric, then the horizontal and complete lifts of a vector and covector fields to $T^*(M_n)$ with metric ${}^{R}\nabla$ are parallel if and only if the given vector and covector fields are parallel relative to the Levi-Civita connection ∇ .

In subchapter 2 of chapter 3 non-Levi-Chivita metric connections and the properties of their curvature tensors are studied.

In subchapter 1 of chapter 3 we introduce ${}^{R}\nabla$ Riemannian extension in $T^*(M_n)$ cotangent bundle space and Levita-Chivita connection ${}^{C}\nabla$ of the metrics ${}^{R}\nabla$ is considered. This is an unique torsion-free affine connection satisfying the equality ${}^{C}\nabla({}^{R}\nabla)=0$. But there is another connection satisfying equality $\tilde{\nabla}({}^{R}\nabla)=0$ and possessing nontrivial torsion tensor. This connection is called a metric connection of the metric ${}^{R}\nabla$.

Theorem 9. The cotangent bundle space $T^*(M_n)$ with the metric connection ${}^{H}\nabla$ has a zero scalar curvature with respect to the metric ${}^{R}\nabla$.

In subchapter 3 of chapter 3 the Killing conditions vector fields with respect to the Riemannian extension in cotangent bundle spaces and the conditions that this metric is a Norden metric are described.

The vector field $X \in \Im_0^1(M_n)$ on the manifold M_n with g pseudo-Riemannian metrics is called a Killing vector field (or infinitesimal isometry) if the relation $L_X g = 0$ is satisfied, where L_X is the Lie derivative.

Theorem 10. For a vector field ${}^{V}\omega$ in the cotagent bundle space with the metrics ${}^{R}\nabla$ to be a Killing vector field, it is necessary and sufficient that the vector field $X^i = g^{ij}\omega_j$ be a Killing vector field.

Let φ be an almost complex structure on M_{2n}. The Kahler-Norden manifold can be determine in the form of the triple (M_{2n}, φ, g) formed by the manifold M_{2n} equipped with g pseudo-Riemannian metrics satisfying the condition $\nabla\varphi = 0$, where ∇ is the Levi-Chivita connection of the metric g and it is assumed that g is a Norden metric.

Since the Levi-Chivita connection ∇ of the metrics g is a torsionless affine connection, we get: If $\Phi_\varphi g = 0$, then φ is integrable. Thus, an almost Norden manifold with conditions $\Phi_{\varphi} g = 0$ and $N_{\varphi} \neq 0$, i.e. an almost holomorphic Norden manifold does not exist.

The horizontal lift ${}^{H}\varphi$ in the cotangent bundle $T^{*}(M_{2n})$ is defined by

$${}^{H}\varphi({}^{V}\omega) = {}^{V}(\omega \circ \varphi), \ \ {}^{H}\varphi({}^{H}X) = {}^{H}(\varphi X)$$

for any vector and covector fields X and ω on $T^{*}(M_{2n})$, respectively. From here it follows that the horizontal lift ${}^{H}\varphi$ has components of the form

$${}^{H}\varphi = (\tilde{\varphi}_{\beta}^{\alpha}) = \begin{pmatrix} \varphi_{j}^{i} & 0 \\ 0 & \varphi_{i}^{j} \end{pmatrix}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, where φ_{j}^{i} are the local components of φ .

It is well known that if φ is an almost complex structure in M_{2n} with torsion free connection ∇, then ${}^{H}\varphi$ is also almost complex structure in $T^{*}(M_{2n})$. Using this fact and the expression of ${}^{R}\nabla$, we can easily show that

$${}^{R}\nabla({}^{H}\varphi\tilde{X}, \tilde{Y}) = {}^{R}\nabla(\tilde{X}, {}^{H}\varphi\tilde{Y})$$

for any $\tilde{X} = {}^{H}X, {}^{V}\omega$ and $\tilde{Y} = {}^{H}Y, {}^{V}\theta$, i.e. the triple $(T^{*}(M_{2n}), {}^{R}\nabla, {}^{H}\varphi)$ is an almost Norden manifold.

The covariant derivative of ${}^{H}\varphi$ with respect to the Levi-Civita connection ${}^{C}\nabla$ of ${}^{R}\nabla$ has components

$${}^{C}\nabla_{i}\, {}^{H}\tilde{\varphi}_{j}^{k} = \nabla_{i}\varphi_{j}^{k}, \quad {}^{C}\nabla_{i}\, {}^{H}\tilde{\varphi}_{\bar{j}}^{\bar{k}} = \nabla_{i}\varphi_{j}^{k},$$

$${}^{C}\nabla_{i}\, {}^{H}\tilde{\varphi}_{j}^{\bar{k}} = \frac{1}{2} p_{a}[(R_{imk}{}^{a} - R_{imk}{}^{a} + R_{kim}{}^{a})\varphi_{j}^{m} - (R_{ijm}{}^{a} - R_{jmi}{}^{a} + R_{mij}{}^{a})\varphi_{k}^{m}]$$

and all other components are equal to zero. If a torsion-free affine connection ∇ preserving the structure φ $(\nabla\varphi = 0)$ satisfies the condition

$$\nabla_{\varphi X} Y = \varphi(\nabla_X Y),$$

then ∇ is called a holomorphic connection. Since the purity of the curvature tensor field of connection ∇ is a necessary and sufficient condition for its holomorphity, from formules of covariant derivative of ${}^H\varphi$ we have

Theorem 11. The cotangent bundle $T^*(M_{2n})$ is a Kähler–Norden manifold with respect to ${}^R\nabla$ and an almost complex structure ${}^H\varphi$ if the torsion-free connection ∇ is a holomorphic connection with respect to the structure φ .

On the other hand, it is well known that the curvature tensor of the Norden metric is pure in a Kähler-Norden manifold. Hence, if M_{2n} has a Kähler-Norden metric g and the Levi-Civita connection ∇ of g and $T^*(M_{2n})$ has the Riemannian extension ${}^R\nabla$ as its metric, then we get the following theorem:

Theorem 12. The cotangent bundle $T^*(M_{2n})$ of a pseudo-Riemannian manifold (M_{2n}, g) is a Kähler–Norden manifold with respect to ${}^R\nabla$ and ${}^H\varphi$ provided that (M_{2n}, g, φ) is a Kähler–Norden manifold.

In subchapter 4 of chapter 3 we determine a new metrics on a contangent bundle space and consider its Levi-Chivita connection.

Let $\widetilde{G}$ be a new metric in the cotangent bundle $T^*(M_n)$:

$$\widetilde{G} = dx^i \delta p_i + \sum_{i,j=1}^{n} g^{ij} \delta p_i \delta p_j .$$

The following theorems are valid.

Theorem 13. If X, Y are parallel vector fields on M_n, then complete lifts of vector fields X, Y to cotangent bundle space are orthogonal with respect to the metric $\widetilde{G}$.

Theorem 14. Vertical lift of the covector field $\omega \in \Im_1^0(M_n)$ to cotangent bundle space with the metric $\tilde{G}$ is not parallel in $T^*(M_n)$.

Teorem 15. The complete and horizontal lifts of the vector field $X \in \Im_0^1(M_n)$ to $T^*(M_n)$ with the metric $\tilde{G}$ are parallel if and only if X is parallel on M_n with respect to ∇.

In subchapter 5 of chapter 3 we study metric connection and geodesic curves with respect to a new netric on a cotangent bundle space.

On the cotangent bundle space $T^*(M_n)$ the Levi-Chivita connection $\tilde{\nabla}$ is a unique torsionless connection satisfying the condition $\tilde{\nabla}\tilde{G} = 0$. But we will find another connection with nontrivial torsion tensor and satisfying the condition $\tilde{\nabla}\tilde{G} = 0$. This connection is called a metric connection of the metrics $\tilde{G}$.

So, we show the validity of the following theorems.

Theorem 16. The cotangent bundle $\left(T^*(M_n), \tilde{G}\right)$ with the metric connection ${}^H\nabla$ has vanishing scalar curvature ${}^H r$ with respect to the metric if and only if the scalar curvature r of ∇_g on M_n is zero.

Theorem 17. Let $\tilde{C}$ be a curve in the cotangent bundle s expressed locally by $x^h = x^h(t),\ p_h = \theta_h(t)$ with respect to the induced coordinates $\left(x^i, x^{\bar{i}}\right) = \left(x^i, p_i\right)$ in $T^*(M_n)$. The curve $\tilde{C}$ is a geodesic of $\tilde{G}$ if it satisfies the following equations:

$$a)\quad \frac{\delta^2 x^h}{dt^2} - \Gamma_{it}^j g^{th} \frac{dx^i}{dt}\frac{\delta p_j}{dt} = 0,$$

$$b)\quad \frac{\delta^2 p^h}{dt^2} + p_m R_{hji}{}^m \frac{dx^i}{dt}\frac{dx^j}{dt} + p_m R_{hit}{}^s g^{tj} \frac{dx^i}{dt}\frac{\delta p_j}{dt} = 0.$$

In subchapter 6 of chapter 3 we consider the Levi-Chivita connection of Sasaki metric in the tensor bundle of type $(0,2)$.

In the bundle $T_2^0(M)$ the Sasaki metric for arbitrary $X,Y \in \Im_0^1(M)$ and $A,B \in \Im_2^0(M)$ is determined by the equalities:

$$
\begin{gathered}
{}^S g\left({}^V A, {}^V B\right) = {}^V\left(G(A,B)\right),\\
{}^S g\left({}^V A, {}^H Y\right) = 0,\\
{}^S g\left({}^H X, {}^H Y\right) = {}^V\left(g(X,Y)\right)
\end{gathered}
$$

Theorem 18. Assume that (M,g) is a Riemannian manifold and ${}^S\nabla$ is a Levi-Chivita connection of the bundle $T_2^0(M)$ possessing the Sasaki metric ${}^S g$. Then for various indices the components ${}^S\Gamma_{IJ}^K$ are calculated by the following formulas:

$$
{}^s\Gamma_{ij}^k = \Gamma_{ij}^k,\ {}^s\Gamma_{\bar i \bar j}^k = {}^s\Gamma_{\bar i j}^{\bar k} = {}^s\Gamma_{\bar i \bar j}^{\bar k} = 0,
$$

$$
{}^s\Gamma_{ij}^k = \frac{1}{2}\left(R_{ijk_1}^m t_{mk_2} + R_{ijk_2}^m t_{k_1 m}\right),
$$

$$
{}^s\Gamma_{i\bar j}^{\bar k} = -\Gamma_{ik_1}^{j_1}\delta_{k_2}^{j_2} - \Gamma_{ik_2}^{j_2}\delta_{k_1}^{j_1},
$$

$$
{}^s\Gamma_{\bar i j}^k = \frac{1}{2}\left(g^{ai_2} t_{sa} R_{.j.}^{ki_1 s} + g^{bi_1} t_{bs} R_{.j.}^{ki_2 s}\right),
$$

$$
{}^s\Gamma_{i\bar j}^k = \frac{1}{2}\left(g^{aj_2} t_{sa} R_{.i.}^{kj_1 s} + g^{bj_1} t_{bs} R_{.i.}^{kj_2 s}\right),
$$

where $R_{.i.}^{kjs} = g^{kl} g^{jm} R_{\lim}{}^s$.

As we know, the Levi-Chivita connection ∇ of the Riemannian metric g for all vector fields $X,Y,Z \in \Im_0^1(M)$ is given by the following Koszul formula:

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) +$$
$$+ g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y).$$

So we get the following result:

Theorem 19. Assume that (M,g) is a Riemannian manifold and ${}^S\nabla$ is the Levi-Chivita connection in the bundle $T_2^0(M)$ possessing Sasaki metris ${}^S g$. Then Levi-Chivita connection ${}^S\nabla$ for all $X,Y,Z \in \Im_0^1(M)$ and $A,B \in \Im_2^0(M)$ satisfies the following relations.

$i)$ $\quad {}^S\nabla_{{}^H X} {}^H Y = {}^H(\nabla_X Y) + \frac{1}{2}(\gamma + \bar{\gamma})R(X,Y),$

$ii)$ $\quad {}^S\nabla_{{}^V A} {}^H Y = \frac{1}{2} {}^H\left(tg^{-1} \circ R(\ ,Y)\tilde{A} + \bar{t}g^{-1} \circ R(\ ,Y)\tilde{A}\right),$

$iii)$ $\quad {}^S\nabla_{{}^H X} {}^H Y = {}^H(\nabla_X B) + \frac{1}{2}\left(tg^{-1} \circ R(\ ,X)\tilde{B} + \bar{t}g^{-1} \circ R(\ ,X)\tilde{B}\right),$

$iv)$ $\quad {}^S\nabla_{{}^V A} {}^V B = 0,$

where $\tilde{A} + g^{i_1 l} g^{i_2 m} A_{lm} = \left(A^{i_1 i_2}\right) \in \Im_0^2(M),$

$$R(\ .Y)\tilde{A} \in T_1^2(M), g^{-1} \circ R(\ ,Y)\tilde{A} \in \Im_0^3(M).$$

Conclusion

1. Interpretation of geodesic curves of Cheeger-Gromoll metric in the tangent bundle were given;
2. It was shown that the complete lift of a symplectic metric in the tangent bundle space was transformed to natural symplectic metric of cotangent bundle space in canonical symplectic mapping;
3. Necessary and sifficient conditions were given for the images of complete lifts of vector, affinor and $(1,2)$ - tensor fields to be again complete lifts;
4. Conditions for vector fields with respect to the extended Riemannian metrics in Peterson’s sense in cotangent bundle to be Killing vector were given and conditions for them to be Norden metric were found.

The main results of the presented thesis has been published in following works:

1. Kazımova, S.F. Kotoxunan laylanma fəzasında Riman metrikasına dair// Əməkdar elm xadimi, akademik Ə.T. Hüseynovun 100 illik yubileyinə həsr olunmuş Elmi konfransın tezisləri. –Bakı: -2007, -s. 81-82.
2. Салимов, А.А., Казимова, С.Ф. Геодезические линии метрики Чигера-Громолла в касательном расслоении//"Riyazi nəzəriyyələr, onların tətbiqi və tədrisi sahəsində olan problemlər" adlı Beynəlxalq Elmi konfrans, -Gəncə: -2008, -s. 94-95.
3. Kazımova, S.F. (2, 0) tipli tenzor laylaşmasında təbii metrikaya dair//Azərbaycanın Ümumilli Lideri Heydər Əliyevin 85 illik yubileyinə həsr olunmuş Tələbə, Magistrant və gənc tədqiqatçıların Respublika Elmi konfransının tezisləri. –Bakı: -2008, -s. 25-26.
4. Salimov, A.A., Kazimova, S.F. Geodesics of the Cheeger-Gromoll metric//Turk. J. Math., -2009, -v. 33, -pp. 99-105 (**SCI-Expanded**).
5. Aslanci, S., Kazimova, S.F., Salimov, A.A. Some notes concerning Riemannian extensions//Ukrainian Math. J., -2010, -v. 62, -№5, -pp. 661-675(**SCI-Expanded**).
6. Kazımova, S.F. Kotoxunan laylanma fəzasında Riman genişlənməsi və onun bəzi tətbiqləri//Azərbaycan xalqının Ümumilli Lideri Heydər Əliyevin anadan olmasının 91 illiyinə həsr olunmuş "Riyaziyyat və mexanikanın aktual problemləri" adlı Respublika Elmi Konfransı. –Bakı: -2014, -12-13 may, -s. 89-92.
7. Фаттаев, Г.Д., Казимова, С.Ф. Риманова метрика в расслоении тензоров типа (1,2) над Римановым многообразием //Bakı Dövlət Universitetinin 95-ci ildönümünə həsr olunmuş "Riyaziyyat və Mexanikanın aktual problemləri"

mövzusunda Elmi Konfransının materialları, -Bakı: -2014,-25-26 dekabr, -s. 49-50.
8. Fəttayev, H.D., Kazımova, S.F. $T_0^3(M)$ laylanması üzərində Riman metrikası və Riman rabitəsi //Azərbaycanın görkəmli alimi və ictimai xadimi, Dövlət mükafatı laureatı, Əməkdar elm xadimi, BDU-nun sabiq rektoru, AMEA-nın müxbir üzvü, professor Y.C.Məmmədovun anadan olmasının 85 illik yubileyinə həsr olunmuş Beynlxalq Elmi Konfransının materialları, -Bakı: -2015, -s. 68-71.
9. Fəttayev, H.D., Kazımova, S.F. Riman çoxobrazlısı üzərində (2,1) tipli tenzorların laylanma fəzasında Riman metrikasına dair // Azərbaycan xalqının Ümumilli Lideri Heydər Əliyevin anadan olmasının 92-ci ildönümünə həsr olunmuş "Riyaziyyat və Mexanikanın aktual problemləri" adlı Respublika Elmi Konfransının materialları,-Bakı: -2015,-20-21 may, -s. 30-32.
10. Фаттаев, Г.Д., Казымова, С.Ф. Риманова метрика в расслоении тензоров типа (1,2) над Римановым многообразием // Вестник Бакинского ун-та, серия физ.-мат. наук, -2017, -№3, -с.58-65.
11. Ocak, F., Kazimova, S.F. On a new metric in the cotangent bundle//Transactions of NAS of Azerbaijan, Issue Mathematics, Series of Phus. – Techn. and Math. Sciences, -2018, -v. 38, -№1, -pp. 128-132.
12. Salimov, A.A., Asl, M.B., Kazimova, S.F. Problems of lifts in symplectic geometry // Chin. Ann. Math., Ser.B, -2019, -v. 40, -№3, -pp. 321-330 (**SCI-Expanded**).
13. Kazimova, S.F. Holomorphic manifolds with deformed lifts of Riemannian metrics // Jour. of Baku Energ. Univer. -2019, -v. 3(1), -pp. 63-68.
14. Казымова, С.Ф. О связности Леви-Чивита метрики Сасаки в тензорном расслоении типа (0,2) // Науч. и педог. извес. Унив. Одлар Юрду, -2019, -№52, -с. 12-19.
15. Salimov, A.A., Kazımova, S.F. Some notes on lifts in symplectic geometry. // Abstracts of the İnternational

Conference Modern Problems of Geometry and Topology and their Applications, -Tashkent, Uzbekistan: -2019, -pp. 20.

The defense will be held on **14 December 2021** year at **12^{00}** at the meeting of the one time Dissertation council BFD 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University.

Address: AZ 1148, Baku city, Z.Khalilov str., 23.

Dissertation is accessible at the library of the Baku State University.

Electronic versions of the dissertation and its abstract are available on the official website of the Baku State University.

Abstract was sent to the required addresses on **11 November 2021**.

Signed for print: 19.10.2021
Paper format: 60x84 1/16
Volume: 33597
Number of hard copies: 20