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ABSTRACT

of the dissertation for the degree of Doctor of Science

SOME PROBLEMS OF FRAME SYSTEMS GENERATED BY BİLİNEAR MAPPİNGS AND THEİR APPLİCATİONS

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the theme and development degree. Bessel property and Hilbert property of a pair of biorthogonal sequences and the Riesz basis in the space L_2 were introduced and studied in 1951 by N.K.Bari[1]. Further there notions were generalized to the to the Banach spaces in various directions in the work of B.E.Weits, Z.A.Chanturia, I.Singer, B.T.Bilalov and Z.G.Huseynov, P.A.Terekhin. Generalization of these notions in the case of the spaces were not studied. In chapter I of there dissertation work we give the notions of Bessel, Hilbert sequences of the Riesz basis in Banach and Hilbert spaces associated with bilinear mappins and alou their uncountable generalizations in nonseparable Banach spaces and give examles.

The Riesz bases are particular cases of frames in Hilbert spaces introduced in 1952 by R.J.Duffin and A.C.Schaeffer[2]. The rapid development of frame theory began in the second half of the eighties after the fundamental works of I.Daubechies, I.Daubechies, A.Grossman and I.Meyer, S.Malat and others. Generalizations of frames in Hilbert spaces with respect to the systems of linear bounded operators were studied by W.Sun. Another generalizations of frames in Hilbert spaces are continuous frames in Hilbert spaces studied in the works of S.T.Ali, J.P.Antoine and J.P.Gazeau, A.Rahimi, A.Najati and Y.N.Dehghan. The notions of Banach frame and atomic expansion in Banach spaces were studied by H.G.Feichtinger and K.Gröchenig. The frames in Banach spaces were studied in the works of O.Christensen, P.G.Casazza, D.Han and D.Larson, B.T.Bilalov, M.R.Abdollahpour, M.H.Faroughi and A.Rahimi, O.Christensen and D.T.Stoeva. Proceeding from the fact that the product of a scalar by a vector

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[1]Bari, N.K. Biorthogonal systems and bases in Hilbert spaces // – Moscow: Ucheniye zapiski MGU, – 1951, 4:148. – c. 69–107.

[2]Duffin, R.J., Schaeffer, A.C. A class of nonharmonic Fourier series // – Washington: Transactions of American Mathematical Society, – 1952. 72, – p. 341–366.

determines bilinear mapping, the study of frames, stability and perturbation bilinear mappings is of great interest. Chapter III of this dissertation work deals with these issues.

Fourier series is one of the main directions of harmonic analysis. Important direction of this theory is to study various properties of Fourier coefficients. In this context, the Housdorff - Young theorem and Hardy-Littlevood theorem are known well. These facts for general orthogonal and uniformly bounded systems in the spaces L_p were studied in the works of F.Riesz and Paley. Many problems of theory of partial equations, of mechanics, mathematical physics and other fields of mathematics are solved by the Fourier method. In this direction, the works of K.I.Khudaveriyev, K.I.Khudaveriyev and A.A.Veliyev, A.Ashyralyev, D.Arjmand, M.Kudu, I.Amirali, J.Nagumo, S.Arimoto, S.Yoshizawa and others are known well. The study of the properties of Fourier coefficients with vector-valued coefficients is of interest. In chapter VI this dissertation work we obtained the analogs of Riesz and Paley theorems in Lebesgue spaces with a mixed norm and in Lebesgue spaces with variable summability index.

One of the important issues in the spectral theory of differential operators is the study of spectral problems with a spectral parameter in boundary conditions. The works of the authors L.Relei, Zh.D. Tamarkin, R.E.Langer, L. Kolatz and others are fundamental results in this direction. The spectral properties of spectral problems with a spectral parameter in the boundary conditions were considered in the works of J. Walter, A. Schneider, C.T. Fulton, D.B. Hinton, H.M. Huseynov etc. General theory of boundary value problems for n-th order ordinary differential equations, when the spectral parameter polynomially included into boundary conditions, was constructed in the paper of A.A.Shkalikov. In the works of E.I.Moiseev and N.Yu.Kapustin considered the basicity of the systems of eigenfunctions in Lebesgue spaces for the Sturm-Liowille problem with a spectral parameter linearly included into the boundary conditions. Various generalizations in this direction were studied in the works of N.B.Kerimov and Z.S.Aliyev, N.B.Kerimov and

V.S.Mirzoev and others. Defective basicity in the abstract form was considered in the work of B.T.Bilalov and T.R.Muradov. In the work of T.B.Gasymov, the criterion with respect to the defective basicity of systems in Banach spaces was found in abstract statement.

Along with this, it should be noted that discontinuous spectral problems with a spectral parameter in the boundary conditions are also of separate scientific interest from in terms of applications. Concerning the related issues, we can consider, for example, the monographs of F.BAtkinson, L.Kollatz and M.A.Rasulov. A spectral problem associated with the problem of oscillation of a loaded string in Lebesgue spaces L_p was studied in the works of T.B.Gasymov and Sh.J. Mammedova, T.B.Gasymov and A.A.Huseynli, B.T.Bilalov, T.B.Gasymov and G.V.Maharramova, and in weighted Lebesgue spaces with a power weight was studied in the work of T.B.Gasymov, A.M.Akhtyamov and N.R.Akhmedzade. It should be noted that the basicity of the systems of eigenfunctions, even in weighted Lebesgue spaces with total weight, apparently have not been studied. In the chapter V of this dissertation work, this problem is considered in grand Lebesgue spaces and in weighted grand Lebesgue spaces with a general form weight.

One of the methods for studying the basic properties of systems is the of perturbation and stability method. Therefore, the study of the basic properties of perturbed systems of exponents in various spaces is of great scientific interest. Note that the basicity of the perturbed system of exponents and trigonometric systems of sines and cosines was studied in the works of A.M.Sedletsky, G.G.Devdariani, E.I.Moiseev, K.I.Babenko, V.F.Gaposhkin, A.N.Barmenkov, A.N.Barmenkov and Yu.A.Kazmin, Yu.I.Lyubarsky, Yu.I.Lyubarsky and V.A.Tkachenko, B.T.Bilalov and others. In many of these papers, the question of completeness and minimality of systems is reduced to the solvability of various Riemann boundary value problems in Hardy classes. Note that the Riemann problem in Lebesgue spaces with variable exponent was studied by B.T.Bilalov and Z.G.Guseynov, and in Morrey spaces in the paper of B.T.Bilalov, T.B.Gasymov and A.A.Guliyeva. In the work of V.M. Kokilashvili, A.Meskhi, V.Paatashvili, the boundary value problem was studied in the subspace of the Lebesgue grand space of functions representable by the Cauchy formula. The basicity of the perturbed system of exponents in grand Lebesgue spaces and the boundary value problem in grand Hardy spaces in the general setting were not considered. The study of the basicity of the perturbed system of exponents in grand Lebesgue spaces by the method of boundary value problems requires the determination of the grand Hardy classes and the solvability of Riemann boundary value problems in them. The chapter VI of the dissertation was devoted to the study of these problem. According to above considerations, we think the topic of the dissertation work is relevant and is of special scientific interest.

Subject and objects of research. The subject and objects of study in the dissertation work are: Bessel, Hilbert systems, Riesz bases and frames in Hilbert and Banach spaces associated with bilinear mappings with respect to the Banach space of sequences of vectors, the Riesz and Paley theorems in Lebesgue spaces and in Lebesgue spaces with variable summability exponent with mixed norm, grand Lebesgue subspaces $G_{p)}$ generated by the shift operator, discontinuous differential operator, grand Hardy classes, solvability of Riemann problems in grand Hardy classes, the perturbed system of exponents.

The goal and tasks of the study. The main goal and tasks of the dissertation work is to obtain various analogs and generalizations of Bessel, Hilbert systems, Riesz bases and frames in Hilbert and Banach spaces, to obtaining the analogs of the Riesz and Paley theorems in Lebesgue spaces with a mixed norm and in Lebesgue spaces with variable summability exponents, to study the basicity of classical systems of exponents and trigonometric systems of sines and cosines in grand Lebesgue subspaces $G_{p)}$ generated by the shift operator, to obtain analogs of Korovkin theorems and their statistical versions in the spaces $G_{p)}$, the basicity of a system of eigenfunctions of one discontinuous spectral problem for a second-order differential equation in $G_{p)}$ and in their weight variants with a general form weight, to determine the grand Hardy classes, to obtain analogs of some classical facts and solvability of the Riemann boundary value problems in grand Hardy classes, and also the basicity of the perturbed system of exponents in grand Lebesgue spaces.

Research methods. The methods of theory of functional analysis, theory of frames, theory of bases, theory of Fourier series, theory of functions, theory of harmonic and complex analysis, theory of partial differential equations, theory of boundary value problems for analytic functions were used in the dissertation work.

The basic aspects to be defended. The following main statements are defended:

1. characterization of Bessel, Hilbert systems, Riesz bases and frames in Hilbert and Banach spaces associated with bilinear mappings;

2. uncountable generalizations of Bessel and Hilbert systems in nonseparable Banach spaces;

3. perturbations and stability of bases and frames associated with bilinear mappings;

4. to obtain analogs and generalizations of the Riesz and Paley theorems in Lebesgue spaces and in Lebesgue spaces with variable summability exponent with mixed norm;

5. the existence and uniqueness of the generalized solution of the mixed problem for one class of third-order differential equations in the space $B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}$ (q,p are conjugate numbers), $p \geq 2$;

6. basicity of the classical systems of exponents and trigonometric systems of sines and cosines in grand Lebesgue subspaces $G_{p)}$ generated by the shift operator;

7. basicity of the system of eigenfunctions of one discontinuous spectral problem for a second-order differential equation in $G_{p)}$ and in weighted versions with a general form weight;

8. to obtain analogs of Korovkin theorems and their statistical variants in $G_{p)}$;

9. determination of grand Hardy classes, to obtain the analogs of the Riesz and Smirnov theorems and to study solvability of Riemann boundary value problems in the grand Hardy classes;

10. to establish basicity of the perturbed system of exponents in grand Lebesgue spaces.

Scientific novelty of the research. In the dissertation work the following main results were obtained:

1. the notions of b-Bessel, b-Hilbert sequences, b-Riesz bases and b-frames in Hilbert and Banach spaces with respect to Banach spaces of sequences of vectors, generalizing the classical notions were introduced and studied their characterization;

2. the notions of an uncountable unconditional basis, uncountable Bessel and Hilbert systems in nonseparable Banach spaces were introduced and analogs of classical results were proved in this case and appropriate examples were given;

3. generalizations of perturbation and stability theorems for bases and frames with respect to b-basis and b-frames in Hilbert and Banach spaces were obtained;

4. the analogs of the Riesz and Paley theorems were found in Lebesgue spaces and in Lebesgue spaces with variable summability exponents with mixed norm, by means of which the existence and uniqueness of a generalized solution of the mixed problem for one class of third-order differential equations in the space $B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}$ (q,p are conjugate numbers), $p \geq 2$ was established;

5. basicity of the classical systems of exponents and trigonometric systems of sines and cosines in the grand Lebesgue subspaces $G_{p)}$ generated by the shift operator, were proved;

6. basicity of the system of eigenfunctions of the differential operator of one discontinuous spectral problem in the direct sum of spaces $G_{p)} \oplus C$, where C is the complex plane, was proved;

7. boundedness of the singular operator was proved in the weighted space $G_{p),\rho}$ in the case when the weight function satisfies the Muckenhoupt condition;

8. basicity of the system of eigenfunctions of one discontinuous spectral problem was proved for a second-order differential equation in a weighted space $G_{p),\rho}$ with a general form weight;

9. grand Hardy classes $H_{p)}$ were determined, the analogs of the Riesz and Smirnov theorems were established, and solvability of Riemann boundary value problems in the grand Hardy classes was studied;

10. the obtained results were applied to the establishment of basicity of the system of exponents with a linear phase in grand Lebesgue subspaces $G_{p)}$.

The theoretical and practical value of the research. The results of the dissertation is mainly of theoretical character. They can be used in the spectral theory of differential operators, in the theory of partial differential equations, in the theory of approximation, in the theory of frames and close bases, in harmonic analysis.

Approbation and application. The results of the dissertation work were reported at the seminar of at the institute seminar of IMM of NASA (head: corr.-member of NASA, prof. M.J.Mardanov), at the seminar of the section "Nonharmonic analysis" of IMM of NASA (head: corr.-member of NASA, prof. B.T.Bilalov), at the seminar of the section "Differential Equations " of IMM of NASA (head: corr.-member of NASA, prof. A.B.Aliyev), at the faculty seminar of the faculty of Mechanics and Mathematics of Baku State University (head: prof. N.Sh.Iskenderov), at the seminar of the department " Theory of functions and functional analysis" of Baku State University (head: prof. A.M.Akhmedov), at the seminar of the department "Mathematical analysis" of Baku State University (head: prof. S.S.Mirzoyev), at the seminar of the department " Differential and integral equations" of Baku State University (head: associate professor Ya.T.Megraliev), at the International conference dedicated to the 85th anniversary of prof. Ya.Dzh.Mammadov (Baku, 2015), at the 12th International Conference on Mathematics and Mechanics, dedicated to the 80th anniversary of Academician F.G.Maksudov (Baku, 2010), at the International Conference dedicated to the 80th the summer anniversary of Academician of ANAS A.D.Hajiyev (Baku, 2017), at the International Conference "Functional Analysis and Its Applications" dedicated to the 100th anniversary of Academician Z.I.Khalilov (Baku, 2011), at the International Conference "Theory of Functions and Problems of Harmonic Analysis" dedicated to the 100th anniversary of Academician I.I.Ibrahimov (Baku, 2012), at the 19th Saratov winter school "Modern problems of the theory of functions and their applications" dedicated to the 90th anniversary of the birth of P.L.Ulyanov (Saratov, 2018), at the International conference "Operators, functions and systems in mathematical physics", dedicated to the 70th anniversary of prof. G.A.Isakhanli (Baku, 2018), at the International Workshop Conference "Nonharmonic Analysis and Differential Equations" (Baku, 2016), at the 2nd International Conference "Mathematical advances and applications" (Istanbul, 2019).

Personal contribution of the author is in formulation of the goal and choice of research direction. Furthermore, all conclusions and the obtained results and research methods belong personally to the author.

Pblications of the author. Publications in editions recommended by HAC under President of the Republic of Azerbaijan – 26, conference materials – 8, abstracts of reports – 2. Of these, 12 articles were published in the journals of the Web of Science database and 1 article in the journal of the Scopus database, which have an impact factor.

Institution where the dissertation work was executed. The work was performed at the Department of Nonharmonic Analysis of the Institute of Mathematics and Mechanics of the Azerbaijan National Academy of Sciences and at the Department of Theory of Functions and Functional Analysis of Baku State University.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately). General volume of the dissertation work consists of – 453739 signs (title page – 424 signs, table of contents – 3315 signs, introduction – 86000 signs, chapter I – 82000 signs, chapter II – 24000 signs, chapter III – 74000 signs, chapter IV – 48000 signs, chapter V – 78000 signs, chapter VI – 58000 signs). Then list of references 227 names.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation consists of an introduction, six chapters and a list of used literature.

Ratioale of the topic of the dissertation work is justified in the introduction, a brief review of concerning matters are given, the basic results of the dissertation are stated.

Chapter I is devoted to a generalization of Bessel, Hilbert sequences and Riesz bases in Hilbert spaces under bilinear mappings. The main results of this chapter were published in the author's works [1-3, 6, 8, 11, 22-25, 29].

In section 1.1 standard denotations, main notions of the theory of bases under bilinear mappings, and also some facts about corcerning the Banach space of sequences of vector sequences are given.

Let X, Y and Z be Banach spaces. Consider a bilinear mapping $b(x,y): X \times Y \to Z$ satisfying the condition $\exists M, m > 0$:

$$m\|x\|_X \|y\|_Y \le \|b(x,y)\|_Z \le M\|x\|_X \|y\|_Y ,\ x \in X ,\ y \in Y .$$

Let $\hat{X}$ be a Banach space of sequences of vectors $\hat{x} = \{x_n\}_{n\in N}$, $x_n \in X$, with coordinatewise linear operations. $\hat{X}$ is said to KB-space if the linear operators $\hat{e}_n : X \to \hat{X}$, $\hat{e}_n(x) = \{\delta_{in} x\}_{i\in N}$, $e_n : \hat{X} \to \hat{X}$, $e_n(\hat{x}) = \{\delta_{in} x_n\}_{i\in N}$, $\hat{\delta}_n : \hat{X} \to X$, $\hat{\delta}_n(\hat{x}) = x_n$,

$\hat{x}=\{x_n\}_{n\in N}\in\hat{X}$, $x\in X$, are bounded. $\hat{X}$ is said to CB-space if subspaces $E_k=\{\hat{x}\in\hat{X}:x_n=0, n\neq k\}$ form a basis in $\hat{X}$.

With respect to space $\hat{X}^*$ it is true

Lemma 1. *Let $\hat{X}$ be a reflexive CB-space. Then the dual space $\hat{X}^*$ is a CB-space.*

In section 1.2 by means of a bilinear map, we give generalizations the concepts of Bessel and Hilbert sequences in Hilbert spaces and prove their corresponding properties.

Let X and H be Hilbert spaces, $\hat{X}$ be a CB-space and $\omega_b : H\times Y\to X$ is bilinear mapping defined by the relation:

$$(b(x,y),h)_H=(x,\omega_b(h,y))_X,\ x\in X,\ h\in H,\ y\in Y.$$

In particular, for $X=C$, $Y=H$ and $b(\lambda,y)=\lambda y$ we have $\omega_b(h,y)=(h,y)_H$.

Definition 1. *The system $\{y_n\}_{n\in N}\subset Y$ and $\{y_n^*\}_{n\in N}\subset Y$ are said b-biorthogonal in H, if $\forall x\in X$*

$$\omega_b(b(x,y_k),y_n^*)=\delta_{nk}x,\ \forall n,k\in N.$$

Let $\{y_n\}_{n\in N}\subset Y$ and $\{y_n^*\}_{n\in N}\subset Y$ be b-biorthogonal in H systems.

Definition 2. *Sequence $\{y_n\}_{n\in N}$ is called b-Bessel in H with respect $\hat{X}$ ($b_{\hat{X}}$-Bessel), if the condition $\{\omega_b(h,y_n^*)\}_{n\in N}\in\hat{X}$ for every $h\in H$, holds.*

The following criterion for $b_{\hat{X}}$-Besselianess holds.

Theorem 1. *In order for the system $\{y_n\}_{n\in N}$ to be $b_{\hat{X}}$-Bessel in H, it is necessary and, in case of b-completeness of $\{y_n\}_{n\in N}$, sufficient that there exist an operator $T\in L(H,\hat{X})$:*

$$T(b(x,y_n))=\{\delta_{in}x\}_{i\in N},\ x\in X, n\in N.$$

Definition 3. *Sequence* $\{y_n\}_{n\in N}$ *is called* b*-Hilbert in* H *with respect* $\hat{X}$ *(*$b_{\hat{X}}$*-Hilbert), if the following condition holds: for* $\forall \hat{x} \in \hat{X}$ $\exists h \in H$ *such that* $\{\omega_b(h, y_n^*)\}_{n\in N} = \hat{x}$.

The following criterion for $b_{\hat{X}}$- Hilbertianess holds.

Theorem 2. *In order for the system* $\{y_n\}_{n\in N}$ *to be* $b_{\hat{X}}$*-Hilbert* H*, it is sufficient and, in case of* b*-completeness of* $\{y_n^*\}_{n\in N}$*, necessary that there exist an operator* $T \in L(\hat{X}, H)$*:*

$$T(\{\delta_{in} x\}_{i\in N}) = b(x, y_n),\ x \in X, n \in N.$$

The following theorem establishes the relation between the $b_{\hat{X}}$-Bessel and $b_{\hat{X}}$-Hilbert systems.

Theorem 3. *Let systems* $\{y_n\}_{n\in N}$ *and* $\{y_n^*\}_{n\in N}$ *be* b*-complete in* H*. The system* $\{y_n\}_{n\in N}$ *is* $b_{\hat{X}}$*-Hilbert in* H *if and only if* $\{y_n\}_{n\in N}$ *is* $b_{\hat{X}^*}$*-Bessel in* H*.*

In section 1.3, using bilinear maps, we generalize Riesz bases in Hilbert spaces, and establish their corresponding relation with Bessel and Hilbert systems.

Definition 4. b*-basis* $\{y_n\}_{n\in N}$ *is called a* $b_{\hat{X}}$*-Riesz basis if its coefficient space coincides with* $\hat{X}$*.*

Let $\{y_n\}_{n\in N} \subset Y$ and $\{y_n^*\}_{n\in N} \subset Y$ be a pair of b-biorthogonal systems.

The following $b_{\hat{X}}$-Riesz basicity criterion is valid.

Theorem 4. *Let* $\{y_n\}_{n\in N}$ *and* $\{y_n^*\}_{n\in N}$ *be* b*-complete in* H*. In order for it* $\{y_n\}_{n\in N}$ *to be a* $b_{\hat{X}}$*-Riesz basis in* H*, it is necessary and sufficient that there exist an boundedly invertible operator* $T \in L(H, \hat{X})$*:*

$$T(b(x, y_n)) = \{\delta_{in} x\}_{i\in N},\ x \in X, n \in N.$$

In section 1.4, proceeding from bilinear mappings, we define the notion of Bessel and Hilbert systems in Banach spaces.

Let $\{y_n\}_{n\in N} \subset Y$ and $\{y_n^*\}_{n\in N} \subset L(Z,X)$ be a pair of b-biorthogonal systems.

Definition 5. *Sequence $\{y_n\}_{n\in N}$ is called b-Bessel in Z with respect $\hat{X}$ ($b_{\hat{X}}$-Bessel), if the condition $\{y_n^*(z)\}_{n\in N} \in \hat{X}$ for $\forall z \in Z$, holds.*

The following criterion for $b_{\hat{X}}$-Besselianess holds.

Theorem 5. *In order for the system $\{y_n\}_{n\in N}$ to be $b_{\hat{X}}$-Bessel Z it is necessary and, in case of b-completeness of $\{y_n\}_{n\in N}$, sufficient that there exist an operator $T \in L(Z,\hat{X})$:*

$$T(b(x,y_n)) = \{\delta_{in}x\}_{i\in N}, \; x \in X, n \in N.$$

Definition 6. *The system $\{y_n\}_{n\in N}$ is called b-Hilbert in Z with respect $\hat{X}$ ($b_{\hat{X}}$-Hilbert), if the following condition holds: for $\forall \hat{x} \in \hat{X}$ $\exists z \in Z$ such that $\{y_n^*(z)\}_{n\in N} = \hat{x}$.*

The following criterion for $b_{\hat{X}}$- Hilbertianess holds.

Theorem 6. *In order for the system $\{y_n\}_{n\in N}$ to be $b_{\hat{X}}$ - Hilbert in Z, it is sufficient and, in case of completeness of $\{y_n^*\}_{n\in N}$, necessary that there exist an operator $T \in L(\hat{X},Z)$:*

$$T(\{\delta_{in}x\}_{i\in N}) = b(x,y_n), \; x \in X, n \in N.$$

We define a mapping $b^*: Z^* \times Y \to X^*$ by the formula

$$b^*(f,y)(x) = f(b(x,y)), \; f \in Z^*, \; y \in Y, \; x \in X.$$

In particular, for $Y = L(Z,X)$ and $b(x,A) = A(x)$, we have $b^*(f,A) = A^*(f)$.

The next theorem establishes a connection between the b-Hilbertianness and the b^*-Besselianness of a pair of b-biorthogonal systems.

Theorem 7. *Let $\hat{X}$ be a reflexive CB-space, Z be a reflexive Banach space, the system $\{y_n\}_{n\in N}$ be b-complete in Z, and the system $\{y_n^*\}_{n\in N}$ be complete in Z^*. Then, in order for $\{y_n\}_{n\in N}$ to be $b_{\hat{X}}$-Hilbert in Z, it is necessary and sufficient that $\{y_n^*\}_{n\in N}$ to be $b_{\hat{X}^*}^*$-Bessel in Z^*.*

In section 1.5, we consider the Bessellianess and Hilbertianess of sequences in Banach spaces without the assumption of the minimality condition.

Let X and Z be Banach spaces, $\hat{X}$ be some KB-space of sequences of vectors of X. Let as consider the system of operators $\{g_k\}_{k\in N} \subset L(Z,X)$.

By $L_{X^*}(\{g_n\}_{n\in N})$ denotes a collection of finite linear combinations of the form $\sum_k x_k^* g_k$, $x_k^* \in X^*$.

Definition 7. *The system $\{g_k\}_{k\in N}$ is called g-complete in Z^*, if $\overline{L_{X^*}(\{g_n\}_{n\in N})} = Z^*$ in the norm Z^*. The systems $\{g_k\}_{k\in N}$ and $\{\Lambda_j\}_{j\in N} \subset L(X,Z)$ are called g-biorthogonal if $g_k \Lambda_j = \delta_{kj} I_X$. The systems $\{g_k\}_{k\in N}$ is called g-minimal in Z^* if $\forall x^* \in X^* \setminus \{0\}$ and $\forall k \in N$ the relation $x^* g_k \notin \overline{L_{X^*}(\{g_n\}_{n\neq k})}$ holds.*

The following notions are generalizations of Bessel and Hilbert systems in Hilbert and Banach spaces.

Definition 8. *The system $\{g_k\}_{k\in N}$ is called $\hat{X}$-Bessel in Z, if the condition $\{g_k(z)\}_{k\in N} \in \hat{X}$ for $\forall z \in Z$, holds. The system $\{g_k\}_{k\in N}$ is called $\hat{X}$-Hilbert in Z, if the following condition holds: for $\forall \{x_k\}_{k\in N} \in \hat{X}$ $\exists z \in Z$ such that $g_k(z) = x_k$.*

The following criterion for $\hat{X}$-Besselianess holds.

Theorem 8. *In order for the system* $\{g_k\}_{k\in N}$ *to be* $\hat{X}$*-Bessel in* Z*, it is necessary and sufficient that there exist an operator* $U \in L(Z,\hat{X})$ *such that* $\hat{\delta}_n U = g_n$ *for every* $n \in N$.

Let us give a criterion for the $\hat{X}$-Hilbert systems.

Theorem 9. *In order for the system* $\{g_k\}_{k\in N}$ *to be* $\hat{X}$*-Hilbert in* Z*, it is sufficient, and in case of* g*-completeness* $\{g_k\}_{k\in N}$ *it is necessary that* $\exists T \in L(\hat{X},Z)$: $g_n T = \hat{\delta}_n$, $\forall n \in N$.

The following theorem establishes the relation between the $\hat{X}$-Bessel and the $\hat{X}$-Hilbert systems.

Theorem 10. *Let* $\hat{X}$ *be a reflexive* CB*-space and* Z *be reflexive. For the system* $\{g_k\}_{k\in N}$ *to be the* $\hat{X}$*-Hilbert system in* Z*, it is sufficient, and in the case of* g*-completeness of* $\{g_k\}_{k\in N}$ *in* Z^* *it is necessary that the following conditions hold:*

1) $\{g_k\}_{k\in N}$ *has the* g*-biorthogonal* $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$*;*

2) the system $\{\Lambda_k^*\}_{k\in N}$ *is* $\hat{X}^*$*-Bessel in* Z^*.

Definition 9. *The system* $\{g_k\}_{k\in N}$ *is called* $\hat{X}^*$*-Riesz* g*-basis in* Z^**, if* $\{g_k\}_{k\in N}$ *is* g*-complete in* Z^* *and there exist* $A>0$ *and* $B>0$ *such that*

$$A\left\|\hat{x}^*\right\|_{\hat{X}^*} \le \left\|\sum_{k=1}^{\infty} x_k^* g_k\right\|_{Z^*} \le B\left\|\hat{x}^*\right\|_{\hat{X}^*},\ \forall \hat{x}^* \in \hat{X}^*.$$

The following theorem is a criterion for $\hat{X}^*$-Riesz g-basicity of systems.

Theorem 11. *Let* $\hat{X}$ *be a reflexive* CB*-space and* Z *be reflexive and the system* $\{g_k\}_{k\in N}$ *is* g*-complete in* Z^**. In order for* $\{g_k\}_{k\in N}$ *to be* $\hat{X}^*$*-Riesz* g*-basis in* Z^**, it is necessary and sufficient that* $\{g_k\}_{k\in N}$ *be both* $\hat{X}$*-Bessel and* $\hat{X}$*-Hilbert system in* Z.

In section 1.6 we give the notion of uncountable Bessel and Hilbert systems in nonseparable Banach spaces and prove the corresponding criteria.

Let X be a nonseparable Banach space, I be some uncountable index set. Let K be a nonseparable Banach space of systems of scalars. Consider the minimal system $\{x_\alpha\}_{\alpha\in I} \subset X$ with the biorthogonal system $\{x_\alpha^*\}_{\alpha\in I} \subset X^*$.

The following concepts are uncountable generalizations of Bessel and Hilbert sequences.

Definition 10. *The system* $\{x_\alpha\}_{\alpha\in I}$ *is called uncountable* K*-Bessel in* X*, if the condition* $\{x_\alpha^*(x)\}_{\alpha\in I} \in K$ *for* $\forall x \in X$*, holds.*

The system $\{x_\alpha\}_{\alpha\in I}$ *is called uncountable* K*-Hilbert in* X*, if the following condition holds: for* $\forall \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ *there exists* $x \in X$ *such that* $\lambda = \{x_\alpha^*(x)\}_{\alpha\in I}$.

Let K be a CB-space with an uncountable unconditional basis $\{\delta_\alpha\}_{\alpha\in I}$ and for $\forall \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in K$ the set $\{\alpha : \lambda_\alpha \neq 0\}$ is at most countable.

Theorem 12. *The system* $\{x_\alpha\}_{\alpha\in I}$ *is uncountable* K*-Bessel in* X*, it is necessary and, in case of completeness of* $\{x_\alpha\}_{\alpha\in I}$ *in* X*, sufficient that there exists an operator* $T \in L(X, K)$ *such that* $Tx_\alpha = \delta_\alpha$*,* $\forall \alpha \in I$.

Theorem 13. *The system* $\{x_\alpha\}_{\alpha\in I}$ *is uncountable* K*-Hilbert in* X *it is sufficient and, in case of completeness of* $\{x_\alpha^*\}_{\alpha\in I}$ *in* X^**, necessary that there exists an operator* $T \in L(K, X)$ *such that* $T\delta_\alpha = x_\alpha$*,* $\forall \alpha \in I$.

Chapter II is devoted to the study of generalizations of results with respect concerning to isomorphic and close bases in Banach spaces in the context of b-basises. The main results of this chapter were published in the author's works [4, 7, 13].

In section 2.1, we study the properties of b-isomorphism of b-basises and the perturbation of b-basises.

Let X, Y and Z be Banach spaces.

In the next theorem, conditions are given for the b-isomorphism to the b-basis under a Fredholm perturbation.

Theorem 14. *Let the system* $\{\varphi_n\}_{n\in N} \subset Y$ *form a* b*-basis in* Z*,* $F \in L(Z)$ *be a Fredholm operator and the system* $\{\psi_n\}_{n\in N} \subset Y$ *such that* $F(b(x,\varphi_n)) = b(x,\psi_n)$*,* $\forall x \in X$*,* $n \in N$*. Then the following properties are equivalent:*

a) $\{\psi_n\}_{n\in N}$ *is* b*-complete in* Z*;*

b) $\{\psi_n\}_{n\in N}$ *is* b*-minimal in* Z*;*

c) $\{\psi_n\}_{n\in N}$ *is* ω*-*b*-linearly independt in* Z*;*

d) $\{\psi_n\}_{n\in N}$ *forms a* b*-basis in* Z*,* b*-isomorphic to* $\{\varphi_n\}_{n\in N}$*.*

Definition 11. *The system* $\{y_n\}_{n\in N}$ *with* b*-biorthigonal system* $\{y_n^*\}_{n\in N}$ *is called* p*-*b*-Bessel in* Z*, if for* $\forall z \in Z$

$$\left(\sum_{n=1}^{\infty} \left\|y_n^*(z)\right\|_X^p\right)^{\frac{1}{p}} \le M_1 \|z\|_Z .$$

If p*-*b*-Bessel in* Z *system* $\{y_n\}_{n\in N}$ *forms a* b*-basis in* Z*, then* $\{y_n\}_{n\in N}$ *is called* p*-*b*-basis in* Z*.*

Theorem 15. *Let* $\{\varphi_n\}_{n\in N} \subset Y$ *form a* p*-*b*-базис for* Z *with* b*-biorthogonal system* $\{\varphi_n^*\}_{n\in N} \subset Q(Z,X)$ *and the system* $\{\psi_n\}_{n\in N} \subset Y$ *is* q*-close to* $\{\varphi_n\}_{n\in N}$*,* $p > 1$*,* $\frac{1}{p} + \frac{1}{q} = 1$*. Then the properties a)-d) of theorem 14 are equivalent.*

In section 2.2 we study the stability of b-bases in Banach spaces.

The following theorem is a generalization of the Riesz basis theorem for a system quadratically close to a Riesz basis.

Theorem 16. *Let* $\{\varphi_n\}_{n\in N}\subset Y$ *forms a* $b_{l_2(X)}$*-Riesz basis for* H*, such that* $\omega_b(\cdot,\varphi_n^*)\in Q(Z,X)$*, the system* $\{\psi_n\}_{n\in N}\subset Y$ *is* ω*-*b*-linearly independent and quadratically close to* $\{\varphi_n\}_{n\in N}$*. Then the system* $\{\psi_n\}_{n\in N}$ *forms a* b*-basis for* H*,* b*-isomorphic to* $\{\varphi_n\}_{n\in N}$*, i.e.* $b_{l_2(X)}$*-Riesz basis.*

The following theorem is a generalization of the Paley-Wiener theorem in Banach spaces.

Theorem 17. *Let the system* $\{\varphi_n\}_{n\in N}\subset Y$ *forms a* b *basis for* Z *and the system* $\{\psi_n\}_{n\in N}\subset Y$ *such that* $\exists\theta\in[0,1)$*, for any finite* $\{x_n\}_{n\in N}\subset X$ *the following relation be valid*

$$\left\|\sum_n b(x_n,\varphi_n-\psi_n)\right\|_Z\le\theta\left\|\sum_n b(x_n,\varphi_n)\right\|_Z.$$

Then the system $\{\psi_n\}_{n\in N}$ *forms a* b*-basis,* b*-isomorphic to* $\{\varphi_n\}_{n\in N}$*.*

In section 2.3 we study a Bessel basis and its stability in a Banach space with respect to the Banach space of sequences of vectors. Analogs of the results of the p-b-basis with respect to the space of sequences of vectors are obtained.

Chapter III is devoted to the study of frames, atomic decompositions and their perturbations in Banach spaces proceeding from bilinear mappings. The main results of this chapter were published in the author's works [9, 12, 14, 16-21, 26-28].

In section 3.1, we given the notions of b-frames and b-frame operators in Hilbert spaces and study some of their properties.

Let X and H be Hilbert spaces and $\{y_n\}_{n\in N}\subset Y$. The next concept is a generalization of frames in Hilbert spaces.

Definition 12. *Sequence* $\{y_n\}_{n\in N}$ *is called* b*-frame in* H*, if there exist* $A,B>0$ *such that*

$$A\|h\|_H^2\le\sum_{k=1}^{\infty}\|\omega_b(h,y_k)\|_X^2\le B\|h\|_H^2,\ \ \forall h\in H. \qquad (1)$$

Constants A and B are called the bounds of b-frame. When the right-hand side of (1) is fulfilled, then the sequence $\{y_n\}_{n\in N}$ is called b-Besselian in H with a bound B.

In the case when $X = C$ and $Y = H$ we have $b(\lambda, y) = \lambda y$, $\omega_b(h, y) = (h, y)_H$ and inequality (1) takes the form

$$A\|h\|_H^2 \le \sum_{k=1}^{\infty} |(h, y_k)|^2 \le B\|h\|_H^2, \ \forall h \in H \ ,$$

i.e. $\{y_n\}_{n\in N}$ is b-frame in H with bounds A and B.

The b-frame criterion holds.

Theorem 18. *Sequence $\{y_n\}_{n\in N}$ forms a b-frame for H if and only if the bounded surjective operator $T : l_2(X) \to Z$, $T(\{x_k\}_{k\in N}) = \sum_{k=1}^{\infty} b(x_k, y_k)$, is defined.*

Let Y_1 be Banach space, H_1 be Hilbert space and $b_1 : X \times Y_1 \to H_1$ be a bounded bilinear mapping.

The next theorem establishes a Noetherian perturbation of b-frames.

Theorem 19. *Let $\{y_n\}_{n\in N}$ be a b-frame in H with bounds A and B, $F \in L(H, H_1)$ be a Noetherian operator, the system $\{\psi_n\}_{n\in N} \subset Y_1$ such that $F(b(x, y_n)) = b_1(x, \psi_n)$ for every $x \in X$, $n \in N$. Then $\{\psi_n\}_{n\in N}$ forms a b-frame for $\overline{L_b(\{\psi_n\}_{n\in N})}$.*

In section 3.2 we study a generalization of frames in Banach spaces by means of bilinear maps in the sense of b-basis.

Definition 13. *The system $\{\varphi_n\}_{n\in N} \subset Y$ is called $b_{\hat{X}^*}$-frame in Z^*, if for any $g \in Z^*$ the condition $\{b^*(g, \varphi_n)\}_{n\in N} \in \hat{X}^*$ is satisfied and there exist constant $A, B > 0$ such that*

$$A\|g\| \le \left\| \{b^*(g, \varphi_n)\}_{n\in N} \right\|_{\hat{X}^*} \le B\|g\|.$$

Let us give a criterion for $b_{\hat{X}^*}$-frames.

Theorem 20. *The system* $\{\varphi_n\}_{n\in N} \subset Y$ $\{\varphi_n\}_{n\in N} \subset Y$ *forms a* $b_{\hat{X}^*}$*-frame for* Z^**, the operator* $T:\hat{X} \to Z$*, defined by formula* $T\hat{x} = \sum_{n=1}^{\infty} b(x_n, \varphi_n)$, $\forall \hat{x} = \{x_n\}_{n\in N} \in \hat{X}$*, is bounded surjective linear operator.*

The next theorem establishes the projection property of $b_{\hat{X}^*}$-frames.

Theorem 21. *Let the system* $\{\varphi_n\}_{n\in N} \subset Y$ *forms a* $b_{\hat{X}^*}$*-frame for* Z^**. Then the following properties are equivalent:*

1) There exist an ambient Banach space Z_1 *space containing* Z *as a closed subspace, which has an* $t_{\hat{X}}$*-basis* $\{\psi_n^*\}_{n\in N} \subset L(X, Z_1)$*, where* $t(x,\psi^*) = \psi^*(x)$*, and* $P \in L(Z_1, Z)$ *is a projector such that* $P\psi_n^*(x) = b(x, \varphi_n)$*, for* $\forall x \in X$*,* $n \in N$*;*

2) $\hat{N} = \left\{ \hat{x} = \{x_n\}_{n\in N} \in \hat{X} : \sum_{n=1}^{\infty} b(x_n, \varphi_n) = 0 \right\}$ *is complemented in* $\hat{X}$*;*

3) there exist $\hat{X}$*-Bessel system* $\{\varphi_n^*\}_{n\in N} \subset L(Z, X)$ *such that* $z = \sum_{n=1}^{\infty} b(\varphi_n^*(z), \varphi_n)$ *for* $\forall z \in Z$.

In section 3.3 we generalize some properties of Banach frames in Banach spaces for the case of spaces of sequences of vectors.

Definition 14. *The system* $\{g_n\}_{n\in N} \subset L(Z, X)$ *is called* $\hat{X}$*-frame in* Z*, there exist constnts* $A, B > 0$ *such that*

$$A\|z\|_Z \le \left\|\{g_k(z)\}_{k\in N}\right\|_{\hat{X}} \le B\|z\|_Z, \ \forall z \in Z.$$

It is valid.

Theorem 22. *Let* $\hat{X}$ *be reflexive* CB*-space, and* Z *be reflexive and* $\{g_n\}_{n\in N} \subset L(Z, X)$*. The system* $\{g_n\}_{n\in N}$ *forms a* $\hat{X}$*-*

frame for Z *only when the operator* $T:\hat{X}^* \to Z^*$, *defined by formula*

$$T(\hat{x}^*) = \sum_{k=1}^{\infty} x_k^* g_k\,,\ \forall \hat{x}^* = \{x_k^*\}_{k\in N} \in \hat{X}^*\,,$$

is bounded, surjective linear operator.

The next concept is a generalization of Banach frames with respect to vector-valued sequences.

Definition 15. *Let* $\{g_n\}_{n\in N} \subset L(Z,X)$ *and* $S:\hat{X}\to Z$ *is a linear operator. The pair* $(\{g_n\}_{n\in N}, S)$ *is called a Banach* $\hat{X}$*-frame in* Z, *if the following conditions are fulfilled*

1) $\{g(z)\}_{n\in N} \in \hat{X}$;

2) there exist $A,B>0$: $A\|z\|_Z \le \|\{g_n(z)\}_{n\in N}\|_{\hat{X}} \le B\|z\|_Z$;

3) S *is bounded and* $S(\{g_n(z)\}_{n\in N}) = z$, $\forall z\in Z$.

It is valid.

Theorem 23. *Let* $\{g_n\}_{n\in N} \subset L(Z,X)$ *be an* $\hat{X}$*-frame in* Z *and the operator* $U\in L(Z,\hat{X})$ *is given by the formula* $U(z) = \{g_n(z)\}_{n\in N}$, $\forall z\in Z$. *The following conditions are equivalent:*

1) $\operatorname{Im} U$ *is complemented in* $\hat{X}$;

2) the operator $U^{-1}:\operatorname{Im} U \to Z$ *can be extended to the bounded operator at all* $\hat{X}$;

3) there exists the bounded operator $S\in L(\hat{X},Z)$ *such that the pair* $(\{g_n\}_{n\in N}, S)$ *forms a Banach* $\hat{X}$*-frame for* Z;

4) $\exists\{\Lambda_n\}_{n\in N} \subset L(X,Z)$ *an* $\hat{X}^*$*-Bessel in* Z^* *such that*

$$z = \sum_{k=1}^{\infty} \Lambda_k g_k(z),\ \forall z\in Z.$$

In section 3.4 we study the relation between $\hat{X}$-frames and $\hat{X}$-Riesz bases in Banach spaces.

Theorem 24. *Let* $\hat{X}$ *be a reflexive* CB*-space and* Z *be reflexive space. Then the following conditions are equivalent:*

1) $\{g_n\}_{n\in N}$ *forms an* $\hat{X}^*$*-Riesz* g*-basis for* Z^* *with the bounds* A *and* B*;*

2) $\{g_n\}_{n\in N}$ *is* $\hat{X}$*-frame in* Z *with the bounds* A *and* B*, and* g*-minimal in* Z^**;*

3) $\{g_n\}_{n\in N}$ *is* g*-complete, be both* $\hat{X}$*-Bessel and* $\hat{X}$*-Hilbert in* Z*.*

In section 3.5 we study the b-atomic decompositions and their Noetherian perturbations in Banach spaces.

Definition 16. *The pair* $\left(\{y_n^*\}_{n\in N}, \{y_n\}_{n\in N}\right)$ *is called* $b_{\hat{X}}$*-atomic decomposition in* Z*, if*

1) $\{y_n^*(z)\}_{n\in N} \in \hat{X}$*;*

2) there exist $A, B > 0$*:* $A\|z\|_Z \le \left\|\{y_n^*(z)\}_{n\in N}\right\|_{\hat{X}} \le B\|z\|_Z$ *;*

3) $z = \sum_{n=1}^{\infty} b(y_n^*(z), y_n)$, $\forall z \in Z$ *.*

If $\{y_n\}_{n\in N}$ *is* $b_{\hat{X}^*}$*-frame in* Z^* *and* $\left(\{y_n^*\}_{n\in N}, \{y_n\}_{n\in N}\right)$ *is* $b_{\hat{X}}$*-atomic decompositions in* Z*, then* $\{y_n\}_{n\in N}$ *is called alternative dual for* $\{y_n^*\}_{n\in N}$.

The next theorem establishes a Noetherian perturbation of the $b_{\hat{X}}$-atomic expansion.

Theorem 25. *Let* $\left(\{y_n^*\}_{n\in N}, \{y_n\}_{n\in N}\right)$ *be a* $b_{\hat{X}}$*-atomic decompositions in* Z *with bounds* A *and* B*,* $F \in L(Z, Z_1)$ *be a Neotherian operator and* $F_b(y_n) = \psi_n$*. Then there exist* $\{\psi_n^*\}_{n\in N} \subset L(Z_1, X)$ *such that* $\left(\{\psi_n^*\}_{n\in N}, \{\psi_n\}_{n\in N}\right)$ *is a* $b_{\hat{X}}$*- atomic decompositions in* $\overline{L_b\left(\{\psi_n\}_{n\in N}\right)}$*. If* $\{y_n\}_{n\in N}$ *is alternative dual* $b_{\hat{X}}^*$*-*

frame in Z^* *for* $\{y_n^*\}_{n\in N}$, *then* $\{\psi_n\}_{n\in N}$ *is alternative dual* $b_{\hat{X}}^*$ *-frame in* $Z_1^* - \ker F^*$ *for* $\{\psi_n^*\}_{n\in N}$.

In Section 3.6 we study the stability of an $b_{\hat{X}}$-atomic decomposition and an $\hat{X}$-frame in Banach spaces.

Let $\{\varphi_n\}_{n\in N} \subset Y$ and $\{\varphi_n^*\}_{n\in N} \subset L(Z, X)$. It is valid

Theorem 26. *Let* $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ *be* $b_{\hat{X}}$ *-atomic decomposition in* Z *with the bounds* A *and* B, *the system* $\{\psi_n\}_{n\in N} \subset Y$. *Assume that there exist the number* $\lambda, \beta, \mu \geq 0$:

i) $\max\{\beta; \lambda + \mu B\} < 1$;

ii) $\left\|\sum_i b(x_i, \varphi_i - \psi_i)\right\|_Z \leq$

$$\leq \lambda \left\|\sum_i b(x_i, \varphi_i)\right\|_Z + \beta \left\|\sum_i b(x_i, \psi_i)\right\|_Z + \mu \left\|\{x_i\}\right\|_{\hat{X}}.$$

Then there exists $\{\psi_n^*\}_{n\in N} \subset L(Z, X)$ *such that* $\left(\{\psi_n^*\}_{n\in N}, \{\psi_n\}_{n\in N}\right)$ *is* $b_{\hat{X}}$ *-atomic decomposition in* Z *with the bounds*

$$\frac{(1-\beta)A}{1+(\lambda+\mu B)} \text{ and } \frac{(1+\beta)B}{1-(\lambda+\mu B)}.$$

In section 3.7, we study similar results on the stability of the Banach $\hat{X}$-frame and the $\hat{X}$-Riess g-basis in Banach spaces.

Chapter IV is devoted to obtaining analogues of the Riesz and Paley theorems in Lebesgue spaces and Lebesgue spaces with variable summability exponent with a mixed norm, and to the study of a generalized solution of the mixed problem for one class of third-order differential equations. The main results of this chapter were published in the author's works [5, 10, 26].

In Section 4.1, we prove analogs of the Riesz and Paley theorems in Lebesgue spaces with mixed norm.

Denote by $l_p(a,b)$, $p>1$, be a Banach space of sequences $a(t)=\{a_i(t)\}_{i\in N}$ of measurable on (a,b) functions, for which the norm

$$\|a\|_{l_p(a,b)}=\left(\sum_{i=1}^{\infty}\int_a^b |a_i(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Let $\{\varphi_n(t)\}_{n\in N}$ be an orthonormed system on $[a,b]$ such that almost everywhere an $[a,b]$ $|\varphi_n(t)|\le M$ $(n\in N)$, M is independent on n.

The following analog of the Riesz theorem[3] in $l_p(a,b)$ hold.

Theorem 27. *The following statements are true:*

1) If $f\in L_{(q,p)}((a,b)\times(c,d))$, $1<p\le 2$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, $a_i(t)=\int_c^d f(t,s)\varphi_i(s)ds$, *then* $a(t)=\{a_i(t)\}_{i\in N}\in l_q(a,b)$, *and*

$$\|a\|_{l_q(a,b)}\le M^{\frac{2-p}{p}}\|f\|_{L_{(q,p)}}$$

2) For any sequence $a(t)=\{a_i(t)\}_{i\in N}\in l_p(a,b)$, $1<p\le 2$, *there exists a function* $f\in L_{(p,q)}((a,b)\times(c,d))$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, *for which* $a_i(t)=\int_c^d f(t,s)\varphi_i(s)ds$, *and such that*

$$\|f\|_{L_{(p,q)}}\le M^{\frac{2-p}{p}}\|a\|_{l_p(a,b)}.$$

In section 4.2, we prove analogs of the Riesz and Paley theorems in Lebesgue spaces with variable summability exponent with a mixed norm.

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[3]Zigmund, A. Triqonometric series: [in 2 vol.] / – Moscow: Mir, – 1965, v.2, – 526 c.

Let Ω and T are measurable sets from R^n, $p(x) \geq 1$, $q(x) \geq 1$ are measurable functions Ω. We denote

$$p_+(E) = ess\sup_{x \in E} p(x) \text{ and } p_-(E) = ess\inf_{x \in E} p(x),$$

in particular, we assume $p_+ = p_+(\Omega)$, $p_- = p_-(\Omega)$. Let $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

Definition 17. *The modular of the measurable function* $f : \Omega \to R$ *with respect* $p(\cdot)$ *is the number*

$$\rho_{p(\cdot),\Omega}(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L_\infty(\Omega_\infty)}.$$

Denote by $L_{p(x)}(\Omega)$ a Banach space of measurable functions $f : \Omega \to R$, for which the norm

$$\|f\|_{L_{p(x)}(\Omega)} = \inf\{\lambda > 0 : \rho_{p(\cdot),\Omega}(f / \lambda) \leq 1\}$$

is finite.

Denote by $L_{q(\cdot)}(\Omega, L_{p(\cdot)}(T))$ a space of measurable on $\Omega \times T$ functions $f(x,t) : \Omega \times T \to R$ such that for almost on Ω $f(x,\cdot) \in L_{p(x)}(T)$ and $\|f(x,\cdot)\|_{L_{p(x)}(T)} \in L_{q(x)}(\Omega)$, and also $l_{p(x)}(\Omega)$ and $l_{p(x),p(x)-2}(\Omega)$ are Banach space of sequences $\{c_k(x)\}_{k \in N}$ of measurable on Ω functions, with corresponding norms

$$\|\{c_k\}\|_{l_{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty} \int_\Omega \left(\frac{|c_k(x)|}{\lambda}\right)^{p(x)} \leq 1\right\},$$

$$\|\{c_k\}\|_{l_{p(x),p(x)-2}(\Omega)} = \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty} \int_\Omega k^{p(x)-2} \left(\frac{|c_k(x)|}{\lambda}\right)^{p(x)} \leq 1\right\}.$$

The following theorems are true.

Theorem 28. *The following statements are true:*

1) If $f \in L_{q(x)}((a,b), L_{p(x)}(c,d))$, $1 < p_- \le p(x) \le 2$, *and* $c_k(x) = \int\limits_c^d f(x,t)\varphi_k(t)dt$, $k \in N$, *then* $\{c_k\}_{k\in N} \in l_{q(x)}(a,b)$ *and*

$$\left\|\{c_k\}_{k\in N}\right\|_{l_{q(x)}(a;b)} \le M_1(p)\|f\|_{L_{q(x)}((a,b),L_{p(x)}(c,d))},\ q(x) = \frac{p(x)}{p(x)-1}.$$

2) For any sequence $\{c_k\}_{k\in N} \in l_{p(x)}(a,b)$, $1 < p(x) \le 2$, *there exists a function* $f \in L_{p(x)}((a,b), L_{q(x)}(c,d))$, *for which* $c_k(x) = \int\limits_c^d f(x,t)\varphi_k(t)dt$, $k \in N$ *and*

$$\|f\|_{L_{p(x)}((a,b),L_{q(x)}(c,d))} \le M_1(p)\left\|\{c_k\}_{k\in N}\right\|_{l_{p(x)}(a,b)},\ q(x) = \frac{p(x)}{p(x)-1}.$$

Theorem 29. *The following statements are true:*

1) If $f \in L_{p(x)}((a,b), L_{p(x)}(c,d))$, $1 < p_- \le p(x) \le 2$, *and* $c_k(x) = \int\limits_c^d f(x,t)\varphi_k(t)dt$, $k \in N$, *then* $\{c_k\}_{k\in N} \in l_{p(x),p(x)-2}(a,b)$ *and*

$$\left\|\{c_k\}_{k\in N}\right\|_{l_{p(x),p(x)-2}(a;b)} \le \frac{AM_1(p)}{p_- - 1}\|f\|_{L_{q(x)}((a,b),L_{p(x)}(c,d))};$$

2) For any sequence $\{c_k\}_{k\in N} \in l_{q(x),q(x)-2}$, $2 \le q(x) \le q_+ < \infty$, *there exists a function* $f \in L_{q(x)}((a,b), L_{q(x)}(c,d))$, *for which* $c_k(x) = \int\limits_c^d f(x,t)\varphi_k(t)dt$, $k \in N$ *and*

$$\|f\|_{L_{q(x)}((a,b),L_{q(x)}(c,d))} \le Aq_+M_2(q)\left\|\{c_k\}_{k\in N}\right\|_{l_{q(x),q(x)-2}(a,b)}.$$

In section 4.3 the existence and uniqueness of a generalized solution to the mixed problem for one class of third-order differential equation is proved.

Let T is a fixed number. By $L_p([0,T],L_{p,p-2}(0,\pi))$, $p\geq 2$, denote a Banach space of function $f(t,x)\in L_p(D)$, $D=(0,T)\times(0,\pi)$, for wich the norm

$$\|f\|_{L_p([0,T],L_{p,p-2}(0,\pi))}=\left(\sum_{n=1}^{\infty}n^{p-2}\int_0^T|f_n(t)|^p dt\right)^{\frac{1}{p}}$$

is finite, where $f_n(t)=\frac{2}{\pi}\int_0^{\pi}f(t,x)\sin nxdx$.

Let $B^{\alpha_0,\alpha_1,..,\alpha_k}_{\beta_0,\beta_1,..,\beta_k,T}$ be Banach space of the function $u(t,x)$ of the form $u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx$, considered in closure of the set D, with $u_n(t)\in C^{(k)}([0,T])$, equipped with the finite norm

$$\|u\|_{B^{\alpha_0,\alpha_1,..,\alpha_k}_{\beta_0,\beta_1,..,\beta_k T}}=\sum_{i=0}^{k}\left(\sum_{n=1}^{\infty}\left(n^{\alpha_i}\max_{0\leq t\leq T}\left|u_n^{(i)}(t)\right|\right)^{\beta_i}\right)^{\frac{1}{\beta_i}},$$

were $\alpha_i\geq 0$, $\beta_i\geq 1$, $i=\overline{0,k}$, $k\geq 0$ is an integer.

Consider the following mixed problem for the equation

$$u_{tt}(t,x)-\alpha u_{txx}(t,x)=F(u)(t,x) \tag{2}$$

with initial and boundary conditions

$$u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x),\ 0\leq x\leq\pi, \tag{3}$$

$$u(t,\pi)=u(t,0)=0,\ 0\leq t\leq T, \tag{4}$$

were $0<\alpha$ is a fixed number, F is an general a given nonlinear operator, φ and ψ are the given functions.

Definition 18. *The function $u(t,x)\in B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ (q,p are are conjugate numbers) satisfying condition (3), is called a generalized solution of (2)-(4), if for any function $v(t,x)\in W_1^1([0,T],L_q(0,\pi))$ such that $v(T,x)=0$ on a.e. $[0,\pi]$, $v(t,0)=v(t,\pi)=0$, $t\in[0,\pi]$, the integral identity*

$$\int_0^T\int_0^\pi \{u_t(t,x)v_t(t,x)-\alpha u_{xx}(t,x)v_t(t,x)+F(u(t,x))v(t,x)\}dxdt-$$

$$-\alpha\int_0^\pi \varphi^{//}(x)v(0,x)dx+\int_0^\pi \psi(x)v(0,x)dx=0.$$

The next theorem proves the existence and uniqueness of a generalized solution to problem (2)-(4).

Theorem 30. *Let the following conditions be satisfied:*

1) $\varphi(x)\in C^{(1)}([0,\pi])\cap W_p^2(0,\pi)$, $\{n^2\varphi_n\}_{n\in N}\in l_{p,p-2}$, $\varphi(0)=\varphi(\pi)=0$, $\psi(x)\in C([0,\pi])\cap W_p^1(0,\pi)$, $\{n\psi_n\}_{n\in N}\in l_{p,p-2}$, $\psi(0)=\psi(\pi)=0$*;*

2) $F(u)\in L_p([0,T],L_{p,p-2}(0,\pi))$, $u\in B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}$, $p\geq 2$*, there exist* $a(t),b(t)\in L_p(0,T)$ *such that*

$$\|F(u)(t,\cdot)\|_{L_{p,p-2}(0,\pi)}\leq a(t)+b(t)\|u\|_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}},\ t\in[0,T];$$

3) there exists $c(t)\in L_p(0,T)$ *:* $\forall u,v\in S(0,R)$

$$\|F(u)(t,\cdot)-F(v)(t,\cdot)\|_{L_{p,p-2}(0,\pi)}\leq c(t)\|u-v\|_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}},\ t\in[0,T],$$

where R *is some number. Then the problem (2)-(4) has a unique generalized solution.*

Chapter V is devoted to the basicity of the classical system of exponents, and also the system of eigenfunctions of one discontinuous spectral problem for a second-order differential equation in grand Lebesgue spaces and in their weighted versions. The main results of this chapter were published in the author's works [31, 33-36].

In section 5.1, we prove the density of the set of continuous functions in the grand Lebesgue subspace generated by the shift operator.

Let $\Omega \subset R^n$ be a measurable set with the finite Lebesgue measure $|\Omega|$ and $p > 1$. Denote by $L_{p)}(\Omega)$ the space of measurable functions f on Ω such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_{\Omega} |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} < +\infty .$$

Consider in $L_{p)}(a,b)$ the shift operator

$$T_{\delta} f(x) = \begin{cases} f(x+\delta), x+\delta \in [a,b], \\ 0, x+\delta \in R \setminus [a,b], \end{cases} , \ \delta > 0 .$$

Let $G_{p)}(a,b)$ be the closure in $L_{p)}(a,b)$ of a linear manifold consisting of the functions $f \in L_{p)}(a,b)$: $\|T_{\delta} f - f\|_{p)} \to 0$, $\delta \to 0$.

The following statement is true.

Theorem 31. *The set* $C_0^{\infty}[a,b]$ *is dense in* $G_{p)}(a,b)$.

In section 5.2, we prove the basicity of the system of exponents and trigonometric systems of sines and cosines in the grand Lebesgue subspace generated by the shift operator.

The basicity of the system of exponents and trigonometric systems of sines and cosines in $G_{p)}(-\pi,\pi)$ is established.

Theorem 32. *Exponential system* $\{e^{\mathrm{int}}\}_{n\in Z}$ *forms a basis for the space* $G_{p)}(-\pi,\pi)$.

Theorem 33. *Systems of sines* $\{\sin nt\}_{n\in N}$ *and cosines* $\{\cos nt\}_{n\in N_0}$ *form bases for the space* $G_{p)}(0,\pi)$.

In section 5.3, we study the basicity of a system of exponents and trigonometric systems of sines and cosines in weighted grand Lebesgue spaces with a general weight.

Let $\rho : [a,b] \to R_+$ be some weight function and $1 < p < +\infty$. By $A_p(a,b)$ we denote the Muckenhoupt class, i. e. the class of weight functions $\rho(t)$ satisfying the condition

$$\sup_{I \subset [a,b]} \frac{1}{|I|} \int_I \rho(t)dt \left(\frac{1}{|I|} \int_I \rho(t)^{-\frac{1}{p-1}} dt \right)^{p-1} < +\infty \cdot$$

Denote by $G_{p),\rho}(a,b)$ the subspace of the space $L_{p),\rho}(a,b)$ of functions f such that $\rho f \in G_{p)}(a,b)$.

The basicity of the classical system of exponents and trigonometric systems in $G_{p),\rho}(-1,1)$ are established.

Theorem 34. *Let the weight function* ρ *belong to the Muckenhope class* $A_p(-1,1)$. *Then the exponential system* $\{e^{in\pi x}\}_{n\in Z}$ *forms basis for the space* $G_{p),\rho}(-1,1)$.

Theorem 35. *Let the weight function* ρ *belong to the Muckenhope class* $A_p(0,1)$. *Then the systems of sines* $\{\sin \pi nx\}_{n\in N}$ *and cosines* $\{\cos \pi nx\}_{n\in Z_+}$ *form bases for the space* $G_{p),\rho}(0,1)$.

In section 5.4, we consider a discontinuous spectral problem for a second-order differential equation in the grand Lebesgue space.

Consider the following discontinuous spectral problem

$$y''(x) + \lambda y(x) = 0, \; x \in (-1,0) \cup (0,1), \tag{5}$$

$$\left.\begin{array}{l} y(-1) = y(1) = 0, \\ y(-0) = y(+0), \\ y'(-0) - y'(+0) = \lambda m y(0), m \neq 0. \end{array}\right\} \tag{6}$$

Define the operator L in $G_{p)}(-1,1) \oplus C$ by the formula

$$L(\hat{u}) = (-u'', u'(-0) - u'(+0))$$

domain $D(L)$, which consists

$$\hat{u} = (u; mu(0)) \in GW^2_{p)}((-1,0) \cup (0,1)) \oplus C$$

satisfying the conditions $u(-1) = u(1) = 0$, $u(-0) = u(+0)$.

As shown in the work of T.B.Gasymov and S.J. Mammadova, problem (5), (6) has the following two series of eigenvalues

$\lambda_{1,n} = (\pi n)^2$, $n = 1,2,...$, $\lambda_{2,n} = \rho_{2,n}^2$, $n = 0,1,2,...$, where $\rho_{2,n}$ it has asymptotics[4]

$$\rho_{2,n} = \pi n + \frac{2}{\pi m n} + O(\frac{1}{n^2})$$

the corresponding eigenfunctions are of the form

$$u_{2n-1}(x) = \sin \pi n x, \; n = 1,2,...,$$

$$u_{2n}(x) = \begin{cases} \sin \rho_{2,n}(1+x), x \in [-1,0] \\ \sin \rho_{2,n}(1-x), x \in [0,1] \end{cases}, \; n = 0,1,2,... .$$

The following theorem is valid.

Theorem 36. *The system of eigenvectors* $\{\hat{u}_n\}_{n \in Z_+}$ *of the operator* L *forms a basis for the space* $G_{p)}(-1,1) \oplus C$, $p > 1$.

In section 5.5, we establish the basicity of the system of eigenfunctions of the differential operator of one discontinuous spectral problem in a weighted grand Lebesgue space with a general weight.

Consider the following discontinuous spectral problem

$$y''(x) + \lambda y(x) = 0, \; x \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1), \qquad (7)$$

$$\left. \begin{array}{l} y(0) = y(1) = 0, \\ y(\frac{1}{3} - 0) = y(\frac{1}{3} + 0), \\ y'(\frac{1}{3} - 0) - y'(\frac{1}{3} + 0) = \lambda m y(\frac{1}{3}), \end{array} \right\} \qquad (8)$$

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[4]Gasymov, T.B., Mammadova, S. J. On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition // – Baku: Transactions of NAS of Azerbaijan, – 2006. 26(4), – p.103–116.

where λ is a spectral parameter, m is a nonzero complex number. Problem (7), (8) has[5] two series of eigenvalues $\lambda_{1,n} = (\rho_{1,n})^2$, $n \in N$, and $\lambda_{2,n} = (\rho_{2,n})^2$, $n \in N \cup \{0\}$, where

$$\rho_{1,n} = 3\pi n\,,\ \rho_{2,n} = \frac{3\pi n}{2} + \frac{2+(-1)^n}{\pi m n} + O(\frac{1}{n^2})\,,$$

the corresponding eigenfunctions are expressed as

$$y_{1,n}(x) = \sin 3\pi n x\,,\ x \in [0,1]\,,\ n \in N\,,$$

$$y_{2,n}(x) = \begin{cases} \sin \rho_{2,n}(x - \frac{1}{3}) + \sin \rho_{2,n}(x + \frac{1}{3}), x \in [0, \frac{1}{3}] \\ \sin \rho_{2,n}(1-x), x \in [\frac{1}{3}, 1] \end{cases},\ n \in N \cup \{0\}.$$

Consider the operator L in the space $G_{p)}(0,1) \oplus C$ by the formula

$$L(\hat{y}) = (-y''; y'(\frac{1}{3} - 0) - y'(\frac{1}{3} + 0))\,,$$

domain $D(L)$, which consists

$$\hat{y} = \left(y; my(\frac{1}{3}) \right) \in GW^2_{p)}((0,\frac{1}{3}) \cup (\frac{1}{3},1)) \oplus C$$

satisfying the conditions

$$y(0) = y(1) = 0\,,\ y(\frac{1}{3} - 0) = y(\frac{1}{3} + 0)\,.$$

Theorem 37. *Let the weight function ρ belong to the class $A_p(0,1)$. Then the system of eigenfunctions $\{\hat{y}_n\}_{n \in Z_+}$ of the operator L forms a bases for space $G_{p),\rho}(0,1) \oplus C$.*

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[5]Gasymov, T.B., Akhtyamov, A.M., Ahmedzade, N.R., On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces // – Baku: Proceeding of IMM of NAS of Azerbaijan, – 2020. v.46, №1, – p. 32–44.

The next theorem considers the basicity of the system of eigenfunctions $\{y_0\} \cup \{y_{i,n}\}_{i=1,2;n\in N}$ of problem (7), (8) in $G_{p),\rho}(0,1)$.

Theorem 38. *Let the weight function ρ belong to the class $A_p(0,1)$. The following statements are true:*

1) if from the system $\{y_0\} \cup \{y_{i,n}\}_{i=1,2;n\in N}$ we eliminate any function $y_{2,n_0}(x)$, corresponding to a simple eigenvalue, then the obtaining system forms a basis for $G_{p),\rho}(0,1)$;

2) if from the system $\{y_0\} \cup \{y_{i,n}\}_{i=1,2;n\in N}$ we eliminate any function $y_{1,n_0}(x)$, then the obtaining system is not complete and is not minimal in $G_{p),\rho}(0,1)$.

In Section 5.6, we study analogs of Korovkin's theorems and their statistical variants in grand Lebesgue spaces.

Theorem 39. *Let $\{L_n\}_{n\in N}$ be a sequence of positive linear operators on $G_{p)}(0,1)$, $p>1$, satisfying the condition*

$$\lim_{n\to\infty} \|L_n g - g\|_\infty = 0, \ \forall g \in \{1, t, t^2\}.$$

Then the relation

$$\lim_{n\to\infty} \|L_n f - f\|_{p)} = 0, \ \forall f \in G_{p)}(0,1),$$

hold if and only if $\sup_n \|L_n\| = c < +\infty$.

Corollary 1. *Let $\{L_n\}_{n\in N}$ be a sequence of positive linear operators on $G_{p)}(0,1)$, $p>1$, such that $\sup_n \|L_n\| = c < +\infty$. If in $C([0,1])$ $\exists st - \lim_{n\to\infty} L_n g = g$, $\forall g \in \{1, t, t^2\}$, then in $G_{p)}(0,1)$ $\exists st - \lim_{n\to\infty} L_n f = f$, $\forall f \in G_{p)}(0,1)$.*

Chapter VI is devoted to the determination of the grand Hardy classes, the solvability of the Riemann problems in the grand Hardy classes, and the basicity of the perturbed system of exponents in the grand Lebesgue spaces. The main results of this chapter were published in the author's works [30, 32].

In Section 6.1, we determine a grand Hardy class, the analogs of the Riesz, Smirnov theorem and the representation of a function by the Cauchy formula are proved.

Let γ be a unit circle $\gamma = \{z \in C : |z| = 1\}$ and $\omega = \operatorname{int}\gamma$. Define the grand Hardy space $H^+_{p)}$, $p > 1$, of functions f analytic in ω which satisfy the condition

$$\|f\|_{H^+_{p)}} = \sup_{0<r<1} \|f_r(\cdot)\|_{p)} < +\infty,\ f_r(t) = f(re^{it}).$$

The first part of the Riesz theorem is valid in $H^+_{p)}$.

Theorem 40. *Every function $f \in H^+_{p)}$, $p > 1$, has boundary values $f^+(\cdot)$ almost everywhere on γ in non-tangential directions, $f^+ \in L_{p)}(0,2\pi)$ and the relation $\|f^+(\cdot)\|_{p)} = \lim_{r\to 1} \|f_r(\cdot)\|_{p)}$* holds.

The second part of the Riesz theorem is true under an additional condition.

Theorem 41. *Let $f \in H^+_{p)}$, $p > 1$. Then the relation $\lim_{r\to 1} \|f_r(\cdot) - f^+(\cdot)\|_{p)} = 0$ holds only when $\lim_{\varepsilon\to +0} \varepsilon \int_0^{2\pi} |f^+(t)|^{p-\varepsilon} dt = 0$.*

In theorem below, Cauchy's formula for the functions from grand Hardy class is obtained.

Theorem 42. *1) If $f \in H^+_{p)}$, $1 < p < +\infty$, then the following Cauchy formula holds:*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^+(\xi)}{\xi - z} d\xi,\ z \in \omega. \tag{9}$$

2) If $f^+ \in L_{p)}(0,2\pi)$, $1 < p < +\infty$, then the function f, defined by (9), belongs to the class $H^+_{p)}$.

In section 6.2, we find a general solution to the homogeneous Riemann problem in the grand Hardy classes.

Let $f(z)$ be an analytic function outside the unit disk ω , and let the Laurent expansion of $f(z)$ in a neighborhood of point at infinity be of the form

$$f(z)=\sum_{k=-\infty}^{m} a_k z^k ,\ z\to\infty .$$

If the regular part $f_0(z)=\sum_{k=-\infty}^{-1} a_k z^k$ is such that $\overline{f_0\left(\frac{1}{z}\right)}\in H^+_{p)}$, $p>1$, then we say that f belongs to the class ${}_mH^-_{p)}$.

Consider the homogeneous Riemann problem in the classes $H^+_{p)}\times{}_mH^-_{p)}$:

$$F^+(\tau)-G(\tau)F^-(\tau)=0,\ \tau\in\gamma, \qquad (10)$$

where $G(\tau)$ is a given measurable function on γ.

Let $\theta(t)=\arg G(e^{it})$, $t\in[-\pi;\pi]$, and $Z_\theta(z)$ is the canonical solution to the homogeneous problem (10).

Theorem 43. *Let the coefficient G of the Riemann problem (10) satisfy the following conditions:*

i) $G^{\pm1}(e^{it})\in L_\infty(-\pi,\pi)$;

ii) $\theta(t)$ *is piecewise Hölder on the closed interval* $[-\pi,\pi]$, *and* $\theta(t)=\theta_0(t)+\theta_1(t)$, *where* $\theta_0(t)$ *is a continuous part of* $\theta(t)$ *and* $\theta_1(t)$ *is the function of jumps of* $\theta(t)$ *at the points of discontinuity of* $-\pi<s_1<s_2<......<s_r<\pi$, *i. e.*

$$\theta_1(-\pi)=0,\ \theta_1(t)=\sum_{k:t<s_k} h_k ,\ t\in(-\pi,\pi],$$

where $h_k=\theta(s_k+0)-\theta(s_k-0)$, $k=\overline{1,r}$.

iii) $-\frac{1}{q}<\frac{h_k}{2\pi}\le\frac{1}{p}$, $k=\overline{0,r}$, $h_0=\theta(-\pi)-\theta(\pi)$.

Then the solvability of problem (10) in classes $H^+_{p)}\times{}_mH^-_{p)}$ *is valid:*

α) for $m \ge 0$, problem (10) has a general solution of the form

$$F(z) = Z_\theta(z) P_k(z),$$

where $P_k(z)$ is an arbitrary polynomial of degree $k \le m$;

β) for $m < 0$ problem (10) has a trivial solution.

In section 6.3, we find a general solution to the inhomogeneous Riemann problem in the grand Hardy classes.

Consider the inhomogeneous Riemann problem in the classes $H^+_{p)} \times_m H^-_{p)}$:

$$F^+(\tau) - G(\tau) F^-(\tau) = f(\tau),\ \tau \in \gamma, \qquad (11)$$

where $f \in L_{p)}(\gamma)$, $p > 1$, $G^{\pm 1}(\tau) \in L_\infty(\gamma)$ are a given measurable function on γ.

The following statement holds.

Theorem 44. *Let the assumptions of Theorem 43 hold and $\frac{h_k}{2\pi} \ne \frac{1}{p}$, $k = \overline{0, r}$. Then the solvability of problem (11) in classes $H^+_{p)} \times_m H^-_{p)}$ is valid:*

α) for $m \ge -1$ problem (11) has a general solution of the form $F(z) = Z_\theta(z) P_k(z) + F_1(z)$, where $P_k(z)$ is an arbitrary polynomial of degree $k \le m$ (for $m = -1$, $P_k(z) = 0$) and $F_1(z)$ is the function defined by formula

$$F_1(z) = \frac{Z_\theta(z)}{2\pi i} \int_\gamma \frac{f(\xi)[Z^+_\theta(\xi)]^{-1}}{\xi - z} d\xi \ ;$$

β) for $m < -1$ problem (11) is solvable if and only if the right-hand side $f \in L_{p)}(\gamma)$ satisfies the orthogonality condition

$$\int_{-\pi}^{\pi} \frac{f(e^{it})}{Z^+(e^{it})} e^{ikt} dt = 0,\ k = \overline{1, -m-1}.$$

In this case, problem (11) has the unique solution $F(z) = F_1(z)$.

In Section 6.4, we prove the boundedness of the singular operator in the weighted subspace $G_{p),\rho}(-\pi,\pi)$ and the basicity of the perturbed system of exponents in $G_{p)}(-\pi,\pi)$, $p>1$.

Let $\rho:[-\pi;\pi]\to R_+$ be some weight function. Consider the identification operator $T:L_{p)}(\gamma)\to L_{p)}(-\pi,\pi)$, defined by the formula $Tf(t):=f(e^{it})$, $t\in[-\pi,\pi]$. Let $G_{p)}(\gamma)$ and $G_{p),\rho}(\gamma)$ be the images of the spaces $G_{p)}(-\pi,\pi)$ and $G_{p),\rho}(-\pi,\pi)$, respectively, during the mapping T^{-1}.

In the next lemma, invariance $G_{p),\rho}(-\pi,\pi)$ with respect to a singular operator S_γ with the Cauchy kernel is proved.

Lemma 2. *Let the weight function* ρ *belong to the Muckenhope class* $A_p(-\pi,\pi)$. *Then* S_γ *acts boundedly in* $G_{p),\rho}(\gamma)$.

The main result of this section is

Theorem 45. *Let* $2\operatorname{Re}\beta+\frac{1}{p}\notin Z$. *Then the system* $E_\beta=\left\{e^{i(n-\beta \operatorname{sign} n)t}\right\}_{n\in Z}$ *forms a basis for* $G_{p)}(-\pi,\pi)$, *if and only if* $\left[2\operatorname{Re}\beta+\frac{1}{p}\right]=0$. *The defect of the system* E_β *is equal to* $d(E_\beta)=\left[2\operatorname{Re}\beta+\frac{1}{p}\right]$. *In other words, when* $d(E_\beta)<0$, *the system* E_β *is not complete, but minimal in* $G_{p)}(-\pi,\pi)$; *when* $d(E_\beta)>0$, *the system* E_β *is complete, but not minimal in* $G_{p)}(-\pi,\pi)$.

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Conclusions

The dissertation work is devoted to the study of Bessel and Hilbert sequences, Riesz bases and frames in Hilbert and Banach spaces with respect to the spaces of sequences of vectors under bilinear mappings, to obtaining the analogs of the Riesz and Paley theorems in Lebesgue spaces with mixed norm and in Lebesgue spaces with variable summability exponents, to establishing a basicity classical systems of exponents and trigonometric systems of sines and cosines in grand Lebesgue subspaces $G_{p)}$ generated by the shift operator, to obtaining analogues of Korovkin's theorems and their statistical variants in spaces $G_{p)}$, to establishing the basicity of a system of eigenfunctions of one discontinuous spectral problem for a second-order differential equation in $G_{p)}$ and in their weighted variants with a general form weight, to determination the grand Hardy classes, to establishing analogues of some classical facts and studying the solvability of Riemann boundary value problems in the grand Hardy classes and also to establishing the basicity properties of a perturbed system of exponents in subspaces $G_{p)}$.

The main results of the dissertation work are the followings:

1. The notions of b-Bessel, b-Hilbert sequences, b-Riesz bases and b-frames in Hilbert and Banach spaces with respect to Banach spaces of sequences of vectors, generalizing the classical notions were introduced and their characterizations was studied;
2. The notions of an uncountable unconditional basis, uncountable Bessel and Hilbert systems in nonseparable Banach spaces are introduced and analogues of classical results in this case are proved;
3. Generalizations of perturbation and stability theorems for bases and frames with respect to b-basises and b-frames in Hilbert and Banach spaces, was obtained;
4. The analogs of the Riesz and Paley theorems was found in Lebesgue spaces with mixed norm and in Lebesgue spaces with variable summability exponent;

5. The existence and uniqueness of the generalized solution of the mixed problem for one class of differential equations of the third order was established in the space $B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}$ (q,p are conjugate numbers), $p \geq 2$;
6. The basicity of the classical systems of exponents and trigonometric systems of sines and cosines in the grand Lebesgue subspaces $G_{p)}$, was proved;
7. The basicity of the system of eigenfunctions of the differential operator of one discontinuous spectral problem was proved in the direct sum $G_{p)} \oplus C$ of spaces, where C is the complex plane;
8. The boundedness of the singular operator in the weighted space $G_{p),\rho}$ was proved in the case when the weight function satisfies the Muckenhoupt condition;
9. The basicity of the system of eigenfunctions of one discontinuous spectral problem for a second-order differential equation was proved in a weighted space $G_{p),\rho}$ with a general form weight;
10. The grand Hardy classes $H_{p)}$ were determined, the analogs of the Riesz and Smirnov theorems were established, and the solvability of Riemann boundary value problems in the grand Hardy classes were studied;
11. The obtained results were applied to establishing of the basicity of the system of exponents with a linear phase in subspaces $G_{p)}$.

The main results of the dissertation work were published in the following works:

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