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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

FACTORIZATION METHOD TO DETERMINATE THE FUNDAMETAL SOUTIONS OF THE FRACTIONAL ORDER ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Specialty: 1211.01-Diffferential equations

Field of science: Mathematics

Applicant: **Allahshukur Azizaga Guliyev**

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The work was performed at the Department of Mathematics and Informatics, Lankaran State University.

Scientific supervisors: Doctor of Sciences in Mathematics, professor
Nihan Alipanah oglu Aliyev
Doctor of Sciences in Mathematics, professor
Natig Sahrab oglu Ibrahimov

Official opponents: Doctor of Physics-Mathematics Sciences, professor
Tahir Sadi oglu Hajiyev

Doctor of Sciences in Mathematics
Ilgar Gurbat oglu Mammadov

Candidate of Physics-Mathematics Sciences, associate professor
Zakir Farman oglu Khankishiyev

Dissertation council FD 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University.

Chairman of the Dissertation council: Full member of ANAS, Doctor of Physics-Mathematics Sciences, professor

Magomed Farman oglu Mekhdiyev

Scientific secretary of the Dissertation council: Doctor of Sciences in Mechanics, associate professor

Laura Faig gizi Fatullayeva

Chairman of the scientific seminar: Doctor of Sciences in Mathematics, professor

Yagub Amiyar oglu Sharifov

GENERAL CHRACTERICS OF THE THESIS

Relevance and elaboration of the topic: It is known that fractional differential equations, which began as a theory of metal memory, are currently applied in many areas. Even in the oil industry, it is more expedient to replace the derivatives in the second order ordinary differential equations by the fractional derivatives. Thus, the order of the first derivative was replaced by 1.82[1]. In this case, the application of the obtained result is considered more satisfactory.

The thesis is devoted to the deriving the fundamental solution to the differential equations of fractional order from the fundamental solution of the ordinary and partial differential equations of integer order by the factorization method.

The fundamental solutions obtained for differential equations with fractional derivatives should allow to consider the analogue of boundary value problems for the integer order ordinary and partial differential equations, considered on N. Aliyev's website[2].

This will determine the relevance of the problems considered in the dissertation, as well as the degree of development.

Aims and objectives of the research: In the dissertation the fundamental solutions of the ordinary and two-dimensional first and second order elliptic type equations are used. Making different factorizations for the operators of these equations the fundamental solutions have been constructed for differential equations of order with both proper and improper fraction. In all cases, analytical expressions were obtained for fundamental solutions.

The goals and objectives of the author of the dissertation are to proof the similar results for fractional differential equations, obtained for the problems for the ordinary and partial differential equations of integer order with non-local boundary conditions. The first stage of

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[1] Aliev, F.A., Aliev, N.A. New inverse problem to determine the order fractional derivatives of the oscillation system // - Baku: The reports of National Academy of sciences of Azerbaijan, physcial-mathematical sciences, - 2019. № 75, - p. 13-16.

[2] Aliyev, N.A. List of publications of Professor Nihan A.Aliyev: [Electronic resource], URL: https://nihan.jsoft.ws/index.php?current=0.

the dissertation is based on the fundamental solutions obtained for various equations.

Those results in the integer order case are given in a whole chapter in the book "Equations of Mathematical Physics" by V.S. Vladimirov.

Research methods: The method used for all problems considered in the thesis (both for ordinary and partial differential equations) is the factorization method. Factorization means descending, taking the smallest from the top, breaking the whole.

In the thesis the analytical expressions are derived for obtaining the fundamental solutions to the fractional order differential equations from the fundamental solution of the both ordinary and partial differential equations by factorization method.

The main results presented for the defense:

- From the fundamental solution of the first-order one-dimensional ordinary differential equation, the fundamental solution of the ordinary one-dimensional equation with the proper fractional order is obtained by factorization method.
- From the fundamental solution of the second order one-dimensional ordinary differential equation the fundamental solution of the ordinary one-dimensional differential equation with improper fraction order is obtained by factorization method.
- From the fundamental solutions of the ordinary differential equations of the first and second order derivatives, analytical expressions are obtained for the fundamental solutions of the ordinary differential equations of fractional order by factorization method.
- From the fundamental solution of the elliptic type Cauchy-Riemann equation of the first order, the fundamental solution of the partial differential equation with proper fraction order is obtained by the factorization method.
- An analytical expression is obtained from the fundamental solution of the two-dimensional Laplace equation for the fundamental solution of the fractional partial differential equation by factorization method.

Scientific novelty of the research: The following main new scientific results were obtained in the thesis.

- Fundamental solutions for ordinary differential equations with proper and improper fraction order are obtained from the fundamental solution of the ordinary differential equation.
- From the fundamental solution of the Cauchy-Riemann equation, the fundamental solution for the partial differential equation with proper and improper fraction order is obtained by the factorization method.
- From the fundamental solution of the Laplace equation is obtained by the factorization method the analytical expression of the fundamental solution for the differential equation with proper and improper fraction order.

Theoretical and practical significance of the research: The results of the thesis are theoretical. They can also be used to obtain an approximate solution in some cases. Thus, this fundamental solution or the approximate value of this solution can be used to obtain the approximate solution to the considered fractional order derivative equation if the domain of the solution does not contain the origin.

Approbation of the thesis and applications. The results obtained in the thesis were reported in the: Republic Scientific Conference "The Problems of Development of the Natural and Humanitarian Sciences" dedicated to the 94th Anniversary of the National Leader Heydar Aliyev (Lankaran, 2017); XXXI International Conference "Problems of Decision Making Under Uncertainties" (2018); "New Stage in the Development of Mathematics" dedicated to the 80th Anniversary of Professor N. Aliyev (2018); Republic Scientific Conference "The ways of Implementation of the in the Education Process" dedicated to the 96th Anniversary of the National Leader Heydar Aliyev (Lankaran, 2019); Republic Scientific Conference "The Power of Unity of the Azerbaijani Nation, Government and Army" dedicated to the 98th Anniversary of the National Leader Heydar Aliyev (Lankaran, 2021);

XXXII International Conference Problems of Decision Making Under Uncertainties (Prague, Czech Republic, 2018), The II International Science Conference "Issues of practice and science" (London, Great Britain, 2021).

Personal impact of the author: All results obtained in the thesis belong to the author. The formulation of the problems considered in the thesis belongs to the supervisors.

Publications: Whole content of the thesis is covered by the 14 published works of the author. The list of the publications is given at the end of the avtoreferat.

The name of the organization where the thesis is done: The thesis is done at the Department of "Mathematics and Informatics" of the Lankaran State University.

The total volume of the thesis with the indication of the separate volumes of the structural units: The thesis consists of (title page - 429 characters, table of contents - 1845 characters) introduction - 46000 characters, Chapter I - 65000 characters, Chapter II - 52000 characters, Chapter III - 55000 characters, Conclusion and the list of 125 references. The total volume of the thesis is 220274 characters.

CONTENT OF THE DISSERTATION

The thesis consists of the introduction, three chapters, conclusion and a list of references. The first chapter consists of six, the second and third chapters, each consisting of three paragraphs. In the introductory part of the work, the relevance and degree of development of the topic, research goals and objectives, research methods, the main results presented for the defense, scientific novelties of the research, theoretical and practical significance of research and similar issues are highlighted. Finally, in the introduction the results of the dissertation are given in a short and concise form.

In the first chapter of the thesis entitled **"Fundamental solutions of the ordinary, linear, constant coefficient differential**

equations with positive order" the fundamental solutions of the first, second and n-th order ordinary differential equations are used. Thus, first the first-order one-dimensional equation, then the second-order one-dimensional equation, and finally the n-th order one-dimensional equations are considered.

From the fundamental solution of the first-order one-dimensional equation, the fundamental solutions of one-dimensional equations with the proper fraction order equation are obtained. Then, from the fundamental solution of the second-order one-dimensional equation, the fundamental solutions of one-dimensional equations with improper faction order equation are obtained by the factorization method. This result was then applied to the n-th-order one-dimensional equation.

The results obtained were continued for the first, second, and n-th order linear, ordinary, linear differential equations with constant coefficients. In all cases, analytical expressions are obtained for the fundamental solutions.

In the first paragraph entitled **"On a construction of the fundamental solution to the one term differential equation with the order less than one"** of the first Chapter using the solution $y(t)=\theta(t)$ of the equation

$$Dy(t)=\delta(t), \tag{1}$$

for the $\frac{1}{2}, \frac{1}{3}, \frac{3}{7}$ and $\alpha \in (0, 1)$ order equation the fundamental solutions are obtained in the form

$$U_1(t)=\frac{t^{-\frac{1}{2}}}{(-\frac{1}{2})!}, \quad U_2(t)=\frac{t^{-\frac{2}{3}}}{(-\frac{2}{3})!}, \quad U_3(t)=\frac{t^{-\frac{4}{7}}}{(-\frac{4}{7})!}, \quad U_4(t)=\frac{t^{\alpha-1}}{(\alpha-1)!}, \tag{2}$$

Theorem 1. The fundamental solution of the one dimensional differential equation with orders $\frac{1}{2}, \frac{1}{3}, \frac{3}{7}$ and $\alpha \in (0, 1)$ are of the form (2) correspondingly.

In the second paragraph entitled **"On a construction of the fundamental solutions to the differential equation with order less**

than 2 and bigger than 1”, using the solution $Y(x) = x\theta(x)$ of the equation

$$D^2 Y(x) = \delta(x), \qquad (3)$$

the fundamental solution of the one dimensional equation of order $\frac{8}{5}, \frac{4}{3}, \frac{11}{7}$ and $\alpha \in (1,\ 2)$ are found in the form

$$Z(x) = \frac{x^{\frac{3}{5}}}{\frac{3}{5}!}, \quad T(x) = \frac{x^{\frac{1}{3}}}{\frac{1}{3}!}, \quad V(x) = \frac{x^{\frac{4}{7}}}{\frac{4}{7}!}, \quad H(x) = \frac{x^{\alpha-1}}{(\alpha-1)!}. \qquad (4)$$

Theorem 2. The fundamental solution of the one dimensional ordinary differential equation of order $\frac{8}{5}, \frac{4}{3}, \frac{11}{7}$ and $\alpha \in (1,\ 2)$ are in the form of (4).

In the third paragraph entitled **“On a construction of the one dimensional ordinary differential equation of order $\alpha \in (n-1,\ \ n)$”** using the fundamental solution

$$Y(x) = \frac{x^{n-1}}{(n-1)!}\theta(x), \qquad (6)$$

of the equation

$$y^{(n)}(x) = f(x), \qquad (5)$$

the formula

$$Z(x) = \frac{x^{\alpha-1}}{(\alpha-1)!}, \qquad (7)$$

is obtained for the fundamental solution of the one dimensional ordinary differential equation of order $\alpha \in (n-1,\ \ n)$.

Theorem 3. The fundamental solution of the one dimensional ordinary differential equation of order $\alpha \in (n-1,\ \ n)$ is of the form (7).

In the fourth paragraph entitled **“On a construction of the fundamental solutions to the linear constant coefficient ordinary differential equations with non-negative order less than one”** the fundamental solutions

$$Y(x) = e^x \theta(x), \qquad (8)$$

of the equation

$$D\, y(x) - y(x) = f(x), \qquad (9)$$

the fundamental solution to the equation

$$D^{\frac{1}{2}} Z(x) - Z(x) = \delta(x), \qquad (10)$$

is constructed in the form

$$Z(x) = \int_0^x \frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^t\, dt + \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} + e^x \theta(x). \qquad (11)$$

Thus the following theorem is proved.

Theorem 4. The fundamental solution of equation (10) is of the form of (11), where $\theta(x)$ is Heavisides's unit function.

The fifth paragraph of the first chapter of the thesis is entitled **"On a construction of the fundamental solutions to the linear, constant coefficient ordinary differential equation with nonnegative order less than two".** Here the fundamental solution

$$Y(x) = \frac{e^{\varphi_2 x} - e^{\varphi_1 x}}{\sqrt{a^2 - 4b}} \theta(x), \qquad (13)$$

of the equation

$$D^2 y(x) + aDy(x) + by(x) = f(x), \qquad (12)$$

is used. Here $x > 0$, a and b are given real numbers,

$$\varphi_k = \frac{-a + (-1)^k \sqrt{a^2 - 4b}}{2}, \qquad k = 1, 2\,, \qquad (14)$$

$\theta(x)$ is Heaviside's unit function.

Applying the factorization method from fundamental solution (13) for the fundamental solution to the equation

$$D^{\frac{3}{2}} Z(x) + \sqrt{\varphi_2} DZ(x) - \varphi_1 D^{\frac{1}{2}} Z(x) - \varphi_1 \sqrt{\varphi_2} Z(x) = \delta(x), \qquad (15)$$

the following formula is obtained

$$Z(x)=\frac{\varphi_2}{\sqrt{a^2-4b}}\int_0^x\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2 t}dt-\frac{\varphi_1}{\sqrt{a^2-4b}}\int_0^x\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1 t}dt-$$

$$-\frac{\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\left[e^{\varphi_2 x}\theta(x)-e^{\varphi_1 x}\theta(x)\right], \qquad (16)$$

Theorem 5. For the fundamental solution of equation (15) formula (16) is valid.

Here by the second method formula (13) is reduced to the form

$$Y(x)=\theta(x)\sum_{k=1}^{\infty}\frac{x^k}{k!}\sum_{m=0}^{k-1}\varphi_2^m\varphi_1^{k-1-m}, \qquad (17)$$

And formula (16) is reduced to

$$Z(x)=\sum_{k=1}^{\infty}\frac{x^{k-\frac{1}{2}}}{(k-\frac{1}{2})!}\sum_{m=0}^{k-1}\varphi_2^m\varphi_1^{k-1-m}-\sqrt{\varphi_2}\sum_{k=1}^{\infty}\frac{x^k}{k!}\sum_{m=0}^{k-1}\varphi_2^m\varphi_1^{k-1-m}, \quad x>0, \qquad (18)$$

Finally, *the last paragraph* of this chapter is entitled as **"On a construction of the fundamental solutions to the linear, constant coefficient ordinary differential equations with nonnegative order smaller than** n $(n\in N)$". Here the fundamental solution

$$Y(x)=\sum_{k=1}^{n}(-1)^{n+k}\frac{W^{(n,k)}}{W}e^{\varphi_k x}\theta(x), \qquad (20)$$

to the equation

$$D^n y(x)-y(x)=f(x), \quad x>0, \qquad (19)$$

is used and applying the factorization method is shown that the fundamental solution of the equation

$$\sum_{s=1}^{2n}D^{n-\frac{s}{2}}Z(x)=\delta(x), \qquad (21)$$

is of the form (when $x>0$)

$$Z(x)=\sum_{k=1}^{n}(-1)^{n+k}\frac{W^{(n,k)}}{W}\varphi_k\int_0^x e^{\varphi_k t}\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}dt-$$

$$-\sum_{k=1}^{n}(-1)^{n+k}\frac{W^{(n,k)}}{W}e^{\varphi_k x}. \qquad (22)$$

Here

$$W(x)=\begin{vmatrix} e^{\varphi_1 x} & e^{\varphi_2 x} & \dots & e^{\varphi_n x} \\ \varphi_1 e^{\varphi_1 x} & \varphi_2 e^{\varphi_2 x} & \dots & \varphi_n e^{\varphi_n x} \\ . & . & \dots & . \\ \varphi_1^{n-1} e^{\varphi_1 x} & \varphi_2^{n-1} e^{\varphi_2 x} & \dots & \varphi_n^{n-1} e^{\varphi_n x} \end{vmatrix}=$$

$$=\begin{vmatrix} 1 & 1 & \dots & 1 \\ \varphi_1 & \varphi_2 & \dots & \varphi_n \\ . & . & \dots & . \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \dots & \varphi_n^{n-1} \end{vmatrix}=W\neq 0, \qquad (23)$$

by $W^{(n,k)}$ is denoted the minor of the element φ_k^{n-1} of determinant (23) standing in the intersection of the n-th line and k-th column and

$$\varphi_k=e^{\frac{2\pi k i}{n}}, \quad k=\overline{1,\ n}. \qquad (24)$$

Theorem 6. The fundamental solution of equation (19) is of the form (20).

The second chapter of the thesis entitled **"On a deriving the fundamental solution to the proper fraction order differential equation using the fundamental solution of the first order elliptic equation by the help of factorization method"** the Cauchy-Riemann equation is considered, that is an elliptic equation of the first order.

Here, first of all, from the fundamental solution of the Cauchy-Riemann equation by the factorization method, an analytical expression for the fundamental solution of the semi-order elliptic type equation is obtained. In this chapter, using the fundamental solution of the Cauchy-Riemann equation on the direction, an analytical expression is obtained for the fundamental solution of the semi-ordered elliptic type equation.

Then, using the obtained above fundamental solution of the Cauchy-Riemann equation, an analytical expression was obtained for the fundamental solution of the $\frac{1}{3}$ order elliptic type equation by the factorization method.

Finally, using the obtained fundamental solution of the Cauchy-Riemann equation, an analytical expression was obtained for the fundamental solution of the $\frac{2}{3}$ order elliptic type equation by the factorization method.

In the first paragraph of this chapter is entitled **"The factorization method to deriving the fundamental solution to the semi-order elliptic type equation from the fundamental solution of the Cauchy-Riemann equation".** Here using the fundamental solution

$$U(x-\xi)=\frac{1}{2\pi}\cdot\frac{1}{x_2-\xi_2+i(x_1-\xi_1)} \tag{25}$$

of the Cauchy-Riemann equation

$$\frac{\partial u(x)}{\partial x_2}+i\frac{\partial u(x)}{\partial x_1}=0 \tag{26}$$

by the help of the factorization

$$(D_2+iD_1)=(D_2^{\frac{1}{2}}+i\sqrt{i}D_1^{\frac{1}{2}})(D_2^{\frac{1}{2}}-i\sqrt{i}D_1^{\frac{1}{2}}) \tag{27}$$

for the fundamental solution of the equation

$$(D_2^{\frac{1}{2}}+i\sqrt{i}D_1^{\frac{1}{2}})V(x)=0 \tag{28}$$

the following formula is obtained

$$V(x)=\frac{1}{4\pi(-\frac{1}{2})!}\cdot\frac{1}{(x_2+ix_1)\sqrt{(x_2+ix_1)}}\times$$

$$\times\ln\frac{(\sqrt{x_2+ix_1}-\sqrt{x_2})(\sqrt{x_1-ix_2}-\sqrt{x_1})}{(\sqrt{x_2+ix_1}+\sqrt{x_2})(\sqrt{x_1-ix_2}+\sqrt{x_1})}+$$

$$+\frac{1}{2\pi(-\frac{1}{2})!\cdot(x_2+ix_1)}\cdot\left(\frac{1}{\sqrt{x_2}}-\frac{i\sqrt{i}}{\sqrt{x_1}}\right). \qquad (29)$$

Here $i=\sqrt{-1}$, $D_k=\frac{d}{dx_k}$, $k=1,\ 2.$

Theorem 7. The fundamental solution of the semi-order elliptic equation (28) is in the form (29).

In this paragraph using the fundamental solution to Cauchy-Riemann equation (26) in the direction x_2

$$U(x)=\theta(x_2)\delta(x_1-ix_2) \qquad (30)$$

B the above method for the fundamental solution of equation (28) the following formula is obtained

$$V(x)=i\frac{(x_2+ix_1)^{-\frac{3}{2}}}{(-\frac{3}{2})!}. \qquad (31)$$

Theorem 8. The fundamental solution of semi-order elliptic type equation (28) is of the form (31).

In the second paragraph entitled **“On a deriving the fundamental solution to the $\frac{1}{3}$ order elliptic type equation from the fundamental solution of the Cauchy-Riemann equation”** using the fundamental solution (26) of Cauchy-Riemann equation (25) and factorization

$$D_2+iD_1=(D_2^{\frac{1}{3}}-iD_1^{\frac{1}{3}})(D_2^{\frac{2}{3}}+iD_1^{\frac{1}{3}}D_2^{\frac{1}{3}}-D_1^{\frac{2}{3}}) \qquad (32)$$

for the fundamental solution of the equation

$$D_2^{\frac{1}{3}}W(x)-iD_1^{\frac{1}{3}}W(x)=\delta(x) \qquad (33)$$

the following formula is derived

$$W(x-\xi)=\frac{1}{\pi(-\frac{2}{3})!}\left\{\frac{1}{\sqrt[3]{(x_2-\xi_2+i(x_1-\xi_1))^5}}\times\right.$$

$$\times\left[\frac{1}{6}\ln\left|\frac{(\sqrt[3]{x_2}-\sqrt[3]{x_2-\xi_2+i(x_1-\xi_1)})^3}{\xi_2-i(x_1-\xi_1)}\right|-\right.$$

$$\left.-\frac{1}{\sqrt{3}}\left[arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2-\xi_2+i(x_1-\xi_1)}}{\sqrt{3}\cdot\sqrt[3]{x_2-\xi_2+i(x_1-\xi_1)}}-\frac{\pi}{6}\right]\right]+$$

$$\left.+\frac{1}{2\sqrt[3]{x_2^2}\,(x_2-\xi_2+i(x_1-\xi_1))}\right\}-\frac{i}{2\pi}\frac{x_1^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{x_2^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{1}{\xi_2+i\xi_1}-$$

$$-\frac{i}{2\pi}\frac{x_1^{-\frac{1}{3}}}{(-\frac{1}{3})!}\int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{dt}{[t-\xi_2-i\xi_1]^2}+$$

$$+\frac{1}{2\pi}\frac{x_2^{-\frac{1}{3}}}{(-\frac{1}{3})!}\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{d\tau}{[-\xi_2+i(\tau-\xi_1)]^2}-$$

$$-\frac{1}{\pi}\int_0^{x_1}\frac{(x_1-\tau)^{-\frac{1}{3}}}{(-\frac{1}{3})!}d\tau\int_0^{x_2}\frac{(x_2-t)^{-\frac{1}{3}}}{(-\frac{1}{3})!}\frac{dt}{[t-\xi_2+i(\tau-\xi_1)]^3}-\frac{i}{\pi(-\frac{2}{3})!}\times$$

$$\times\left\{-\frac{1}{\sqrt[3]{(x_1-\xi_1-i(x_2-\xi_2))^5}}\left[\frac{1}{6}\ln\left|\frac{\left(\sqrt[3]{x_1}-\sqrt[3]{x_1-\xi_1-i(x_2-\xi_2)}\right)^3}{\xi_1+i(x_2-\xi_2)}\right|-\right.\right.$$

$$\left.-\frac{1}{\sqrt{3}}\left(arctg\frac{2\sqrt[3]{x_1}+\sqrt[3]{x_1-\xi_1-i(x_2-\xi_2)}}{\sqrt{3}\,\sqrt[3]{x_1-\xi_1-i(x_2-\xi_2)}}-\frac{\pi}{6}\right)\right]+$$

$$\left.+\frac{1}{2\sqrt[3]{x_1^2}\,\sqrt[3]{(x_1-\xi_1-i(x_2-\xi_2))^2}}\right\}. \qquad (34)$$

Thus the following theorem is proved.

Theorem 9. For the fundamental solution to equation (33) formula (34) is valid.

Then in this paragraph using the fundamental solution of the Cauchy-Riemann equation in the direction x_2 for the fundamental solution of equation (33) the following formula is obtained

$$W(x) = \frac{3}{2} \cdot \frac{(x_1 - ix_2)^{-\frac{5}{3}}}{(-\frac{5}{3})!}. \qquad (35)$$

Theorem 10. For the fundamental solution of semi-order elliptic type equation (33) is of the form (35).

In the last paragraph entitled **"Deriving the fundamental solution for the two dimensional, linear, $\frac{2}{3}$ order elliptic equation from the fundamental solution of the Cauchy-Riemann equation"** using the fundamental solution (26) of equation (25) by the help of the factorization

$$(D_2 + iD_1) = (D_2^{\frac{2}{3}} + iD_1^{\frac{1}{3}} D_2^{\frac{1}{3}} - D_1^{\frac{2}{3}})(D_2^{\frac{1}{3}} - iD_1^{\frac{1}{3}}) = \delta(x) \qquad (36)$$

for the fundamental solution of the equation

$$D_2^{\frac{2}{3}} V(x) + iD_1^{\frac{1}{3}} D_2^{\frac{1}{3}} V(x) - D_1^{\frac{2}{3}} V(x) = \delta(x) \qquad (37)$$

the following formula is obtained

$$\frac{1}{2\pi(-\frac{1}{3})!}\left\{-\frac{1}{3(x_1 - ix_2)\sqrt[3]{x_1 - ix_2}}\left[\frac{1}{2}\ln\frac{\left(\sqrt[3]{x_1 - ix_2} - \sqrt[3]{x_1}\right)^3}{-ix_2} + \right.\right.$$

$$\left.\left. + \sqrt{3}(arctg\frac{2\sqrt[3]{x_1} + \sqrt[3]{x_1 - ix_2}}{\sqrt{3}\sqrt[3]{x_1 - ix_2}} - \frac{\pi}{6})\right] - \frac{1}{\sqrt[3]{x_1} \cdot (x_1 - ix_2)}\right\}. \qquad (38)$$

Theorem 11. The fundamental solution of the two dimensional, linear, $\frac{2}{3}$ order elliptic equation is in the form (38).

In the last third chapter of the thesis entitled **"On a deriving the fundamental solution to the fractional order equation from the fundamental solution of the second order Laplace equation by the factorization method"** the second order Laplace equation is considered.

In this chapter first an analytic expression is obtained for the fundamental solution of the $\frac{3}{2}$ order elliptic equation by factorizing the Laplace equation. In the $\frac{3}{2}$ order equation the derivatives change by $\frac{1}{2}$ step. The sum of the orders of derivatives are equal to $\frac{3}{2}$.

Then the operator of the Laplace operator is changed by such way that an analytic expression is obtained for the fundamental solution of the elliptic equation of order $\frac{4}{3}$. In the obtained equation the sum of the orders of derivatives in each term is equal to $\frac{4}{3}$.

Finally the operator of the Laplace equation is factorized and an analytic expression is obtained for the fundamental solution for the $\frac{5}{3}$ order linear constant coefficient elliptic equation by $\frac{1}{3}$ step. Here the sum of the order of derivatives in each term is also equal to $\frac{5}{3}$.

In the first paragraph entitled **"Deriving the fundamental solution for the $\frac{3}{2}$ order equation from the two dimensional Laplace equation applying factorization method"** using the fundamental solution

$$U(x) = \frac{1}{2\pi} \cdot \ln|x| , \qquad |x| = \sqrt{x_1^2 + x_2^2} \tag{39}$$

of the two dimensional Laplace equation

$$\Delta U(x) \equiv \frac{\partial^2 U(x)}{\partial x_2^2} + \frac{\partial^2 U(x)}{\partial x_1^2} = 0 \tag{40}$$

By the help of the factorization

$$D_2^2 + D_1^2 = (D_2^{\frac{1}{2}} + \sqrt{i} D_1^{\frac{1}{2}})(D_2^{\frac{3}{2}} - \sqrt{i} D_2 D_1^{\frac{1}{2}} + i D_2^{\frac{1}{2}} D_1 - i\sqrt{i} D_1^{\frac{3}{2}}) \tag{41}$$

or

$$D_2^2 + D_1^2 = (D_2^{\frac{3}{2}} - \sqrt{i}D_2 D_1^{\frac{1}{2}} + iD_2^{\frac{1}{2}} D_1 - i\sqrt{i}D_1^{\frac{3}{2}})(D_2^{\frac{1}{2}} + \sqrt{i}D_1^{\frac{1}{2}}) \quad (42)$$

for the fundamental solution of the equation

$$D_2^{\frac{3}{2}}V(x) - \sqrt{i}D_2 D_1^{\frac{1}{2}}V(x) + iD_2^{\frac{1}{2}} D_1 V(x) - i\sqrt{i}D_1^{\frac{3}{2}}V(x) = \delta(x) \quad (43)$$

the following formula is obtained

$$V(x) = \frac{1}{2\pi(-\frac{1}{2})!}\left\{ x_2^{-\frac{1}{2}} \ln x_1 - \frac{1}{2}\left[\frac{1}{\sqrt{x_2 + ix_1}} \ln\left| \frac{\left(\sqrt{x_2 + ix_1} - \sqrt{x_2}\right)^2}{ix_1} \right| + \right.\right.$$

$$\left. + \frac{1}{\sqrt{x_2 - ix_1}} \ln\left| \frac{\left(\sqrt{x_2 - ix_1} - \sqrt{x_2}\right)^2}{-ix_1} \right| \right] + \sqrt{i}\left[x_1^{-\frac{1}{2}} \ln x_2 - \right.$$

$$\left.\left. -\frac{1}{2}\left[\frac{1}{\sqrt{x_1 + ix_2}} \ln\left| \frac{\left(\sqrt{x_1 + ix_2} - \sqrt{x_1}\right)^2}{ix_2} \right| + \frac{1}{\sqrt{x_1 - ix_2}} \ln\left| \frac{\left(\sqrt{x_1 - ix_2} - \sqrt{x_1}\right)^2}{-ix_2} \right| \right]\right]\right\}. \quad (44)$$

Theorem 12. The fundamental solution of equation (43) is in the form (44).

In the second paragraph entitled **“On a deriving the fundamental solution to the $\frac{4}{3}$ order equation from the fundamental solution of the two dimensional Laplace equation by factorization method”** from fundamental solution (39) of equation (40) by the help of the factorization

$$D_2^2 + D_1^2 = (D_2^{\frac{2}{3}} + D_1^{\frac{2}{3}})(D_2^{\frac{4}{3}} - D_1^{\frac{2}{3}} D_2^{\frac{2}{3}} + D_1^{\frac{4}{3}}) \quad (45)$$

or

$$D_2^2 + D_1^2 = (D_2^{\frac{4}{3}} - D_1^{\frac{2}{3}} D_2^{\frac{2}{3}} + D_1^{\frac{4}{3}})(D_2^{\frac{2}{3}} + D_1^{\frac{2}{3}}) \quad (46)$$

For the fundamental solution of the equation

$$D_2^{\frac{4}{3}}V(x) - D_1^{\frac{2}{3}} D_2^{\frac{2}{3}}V(x) + D_1^{\frac{4}{3}}V(x) = \delta(x) \quad (47)$$

the following formula is obtained

$$V(x)=\frac{1}{2\pi(-\frac{2}{3})!}\Bigg\{x_2^{-\frac{2}{3}}\ln x_1-\frac{1}{2}\frac{1}{\sqrt[3]{(x_2+ix_1)^2}}\times$$

$$\times\left[\frac{1}{2}\ln\left|\frac{\left(\sqrt[3]{x_2+ix_1}-\sqrt[3]{x_2}\right)^3}{ix_1}\right|-\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2+ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2+ix_1}}-\frac{\pi}{6}\right)\right]-$$

$$-\frac{1}{2}\frac{1}{\sqrt[3]{(x_2-ix_1)^2}}\left[\frac{1}{2}\ln\left|\frac{\left(\sqrt[3]{x_2-ix_1}-\sqrt[3]{x_2}\right)^3}{-ix_1}\right|-\right.$$

$$\left.-\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2}+\sqrt[3]{x_2-ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2-ix_1}}-\frac{\pi}{6}\right)\right]+x_1^{-\frac{2}{3}}\ln|x_2|-$$

$$-\frac{1}{2}\frac{1}{\sqrt[3]{(x_1+ix_2)^2}}\left[\frac{1}{2}\ln\left|\frac{\left(\sqrt[3]{x_1+ix_2}-\sqrt[3]{x_1}\right)^3}{ix_2}\right|-\right.$$

$$\left.-\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1}+\sqrt[3]{x_1+ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1+ix_2}}-\frac{\pi}{6}\right)\right]-$$

$$-\frac{1}{2}\frac{1}{\sqrt[3]{(x_1-ix_2)^2}}\left[\frac{1}{2}\ln\left|\frac{\left(\sqrt[3]{x_1-ix_2}-\sqrt[3]{x_2}\right)^3}{-ix_2}\right|-\right.$$

$$\left.-\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1}+\sqrt[3]{x_1-ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1-ix_2}}-\frac{\pi}{6}\right)\right]\Bigg\}. \qquad (48)$$

Theorem 13. The solution of the $\frac{4}{3}$ order homogeneous constant coefficient partial differential equation (47) is in the form (48).

Finally, *in the last paragraph* entitled **"Deriving the fundamental solution for the $\frac{5}{3}$ order partial differential equation by $\frac{1}{3}$ step from the fundamental solution of the Laplace**

equation by factorization method” from fundamental solution (39) of equation (40) applying the factorization

$$D_2^2 + D_1^2 = (D_2^{\frac{5}{3}} - iD_2^{\frac{4}{3}}D_1^{\frac{1}{3}} - D_2 D_1^{\frac{2}{3}} + iD_2^{\frac{2}{3}}D_1 + D_2^{\frac{1}{3}}D_1^{\frac{4}{3}} - iD_1^{\frac{5}{3}})(D_2^{\frac{1}{3}} + iD_1^{\frac{1}{3}}) \quad (49)$$

for the solution of the equation

$$D_2^{\frac{5}{3}}Z(x) - iD_2^{\frac{4}{3}}D_1^{\frac{1}{3}}Z(x) - D_2 D_1^{\frac{2}{3}}Z(x) + iD_2^{\frac{2}{3}}D_1 Z(x) + \\ + D_2^{\frac{1}{3}}D_1^{\frac{4}{3}}Z(x) - iD_1^{\frac{5}{3}}Z(x) = \delta(x) \quad (50)$$

the following formula is derived

$$Z(x) = \frac{1}{2\pi(-\frac{1}{3})!}\left\{ x_2^{-\frac{1}{3}} \ln|x_1| - \frac{1}{4}\left[\frac{1}{\sqrt[3]{x_2 + ix_1}} \times \right.\right.$$

$$\times\left(\ln\left|\frac{(\sqrt[3]{x_2 + ix_1} - \sqrt[3]{x_2})^3}{ix_1}\right| + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2} + \sqrt[3]{x_2 + ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2 + ix_1}} - \frac{\pi}{6}\right)\right) +$$

$$\left. + \frac{1}{\sqrt[3]{x_2 - ix_1}}\left(\ln\left|\frac{(\sqrt[3]{x_2 - ix_1} - \sqrt[3]{x_2})^3}{-ix_1}\right| + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_2} + \sqrt[3]{x_2 - ix_1}}{\sqrt{3}\cdot\sqrt[3]{x_2 - ix_1}} - \frac{\pi}{6}\right)\right)\right] +$$

$$+ ix_1^{-\frac{1}{3}} \ln x_2 - \frac{i}{4}\left[\frac{1}{\sqrt[3]{x_1 + ix_2}}\left(\ln\left|\frac{(\sqrt[3]{x_1 + ix_2} - \sqrt[3]{x_1})^3}{ix_2}\right| + \right.\right.$$

$$\left. + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1 + ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1 + ix_2}} - \frac{\pi}{6}\right)\right) + \frac{1}{\sqrt[3]{x_1 - ix_2}}\left(\ln\left|\frac{(\sqrt[3]{x_1 - ix_2} - \sqrt[3]{x_2})^3}{-ix_2} + \right| \right.$$

$$\left.\left.\left. + 2\sqrt{3}\left(arctg\frac{2\cdot\sqrt[3]{x_1} + \sqrt[3]{x_1 - ix_2}}{\sqrt{3}\cdot\sqrt[3]{x_1 - ix_2}} - \frac{\pi}{6}\right)\right)\right]\right\}. \quad (51)$$

Theorem 14. The fundamental solution of the $\frac{5}{3}$ order, homogeneous constant coefficient partial differential equation (50) is in the form (51).

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CONCLUSION

The following new results are obtained in the thesis:

1. Fundamental solutions are obtained for the ordinary one-dimensional differential equations of proper and improper fraction orders (i.e., with orders from (0, 1) and (1, 2)). Here, the fundamental solution of the first-order one-dimensional differential equation is factorized.
2. The above results were obtained for the constant coefficient linear, fractional order differential equations. Here, the fundamental solution of the second-order one-dimensional differential equation is factorized.
3. The results obtained in the first paragraph are obtained for the non-one dimensional, linear, constant coefficient fractional equations. Here, the fundamental solution of the first-order differential equation with constant coefficients is factorized.
4. From the fundamental solution of the ordinary, constant coefficient, linear, second order differential equation by factorization method, the fundamental solution of the ordinary, constant coefficient, linear, fractional differential equation with proper and improper fraction order is obtained.
5. From the fundamental solution of the Cauchy-Riemann equation by the factorization method the expression is obtained for the fundamental solution of the partial differential equation with the proper and improper fraction order.
6. From the fundamental solution of the two-dimensional Laplace equation by the factorization method, the analytical expression is obtained for the fundamental solution of the partial differential equation of improper fraction order.

7. The solution of boundary-value problems with the help of obtained here fundamental solutions will then be studied.

The main results of the thesis are published in the following works:

1. Guliyev, A. Fundamental solution of the semi-order elliptic type equation // Republican Scientific Conference on "Problems of development of natural and humanitarian sciences", - Lankaran: - May 5-6, 2017, - pp.39-41.
2. Aliyev, N., Guliyev, A. Factorization method to obtain the fundamental solution of the semi-order elliptic type equation from the fundamental solution of the Cauchy-Riemann equation // Lankaran State University, Scientific news, Natural sciences series, - 2017, No.2, - pp.67-72.
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Address: Z. Khalilov, 23, AZ1148, Baku.

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