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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

STUDYING SPECTRAL PROPERTIES OF SOME CLASS OF DIFFERENTIAL OPERATORS

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GENERAL CHARACTERISTIKS OF WORK

Rationale of the topic and development degree. Spectral theory of operator-differential equations is one of the rapidly developing fields of modern mathematics. Spectral properties of operator-differential equations have been studying from the sixties of the past century. F.S.Rofe-Beketov has proved expansion theorems on eigen functions for not self-adjoint Sturm-Liouville equation with on operator coefficient. Then, F.Z.Ziyetdinov, M.L.Gorbachuk, M.G.Gimadislamov, R.Z.Khalilova, M.G.Gasymov, V.V.Jikov and B.M.Levitan, D.R.Yafayev and others have obtained valuable results in this direction.

Reducing the Schrodinger equation given in $(2n+1)$ odd-dimensional Euclidean space to an unbounded coefficient operator equation, M.G.Gasymov has studied the structure of the spectrum and spectral expansion of the given operator.

T.Kato’s work devoted to the study of spectral properties of many-dimensional Schrodinger operator should be especially underlined.

A.G.Kostyuchenko and B.M.Levitan have determined discreteness condition of the spectrum of unbounded, not self-adjoint Sturm-Liouville type equations with an operator coefficient and obtained the asymptotic formula of distribution function of eigen-values.

B.M.Levitan has comprehensively studied the Green function of self-adjoint Sturm-Liouville operator with and operator coefficient. To this end, he has introduced Banach spaces consisting of operator-valued functions and studied the integral equation of the Green function and its other properties in these spaces. As was noted, these two scientific papers have led to numerous srudies. Valuable studies in this field were conducted both in the fomer Soviet Union and in our Republic. In this field we can mention the work of E.Abdukadirov, E.A.Begovatov, K.Kh.Boymatov, M.Bayramoglu, V.I.Gorbachuk, V.I.Gorbachuk and M.L.Gorbachuk, V.I.Bruk, A.N.Kochubey, V.A.Mikhaylets, V.A.Kutovoy, B.M.Levitan and

G.A.Suvorchenkova, V.P.Maslov, M.Otelbayev, H.I.Aslanov, A.A.Abudov and H.I.Aslanov, H.D.Orujov, B.A.Aliyev, N.M.Aslanova, A.A.Adigezalov, B.A.Aliyev, A.M.Bayramov and others.

Detailed bibliograpy of these works are in the works by M.Sh.Birman and M.Z.Solomyak, R.A.Alexandryan, Y.M.Berezanskiy, V.I.Ilin and in the monograph of A.G.Kostyuchenko and I.S.Sargsyan.

The results obtained by B.M.Levitan were generalized and developed by M.Bayramoglu, H.I.Aslanov, A.A.Abudov, B.I.Aliyev and some followers of M.Bayramoglu and H.I.Aslanov.

Afterwards, G.I.Mishnayevskiy, B.I.Aliyev, M.G.Dushdurov and others have studied the Green function for normal operator coefficient differnetial equations. One of the important issues of the spectral theory of differential operators is the study of discreteness conditions of the negative spectrum of the given operator. Negative spectrum of scalar differnetial operators were studied by N.Roselfeld, B,Y,Skachek, Q.I.Rosenbloom, R.V.Huseynov and others.

The discretness of the negative spectrum and asymptotic distribution of operator-differential equations were studied by M.G.Gasymov, V.V.Jikov and B.M.Levitan, D.R.Yafayev, M.Bayramoglu, A.D.Adigezalov, G.A.Zeynalov, A.M.Bayramov and others. One of the spectral problems of differential operators is to study the resolvent of the given operator. M.Otelbayev has determined the conditions on the inclusion of the resolvent of the operator coeffcients Sturm-Liouville equation in different σ_p Neumann-Shatten classes. Inclusion of the spectrum of higher order operator differential equations into various σ_p classes was proved by R.Kh.Boymatov. M.G.Dushdurov, H.I.Aslanov and G.I.Gasymova have proved that the resolvent of second order operator-differential equations on a finite segment is included into the class of Hilbert-Schmidt type operators.H.I.Aslanov and N.S.Abdullayeva have shown that the resolvent of higher order operator-differential equations on a finite segment is included into the σ_2 Hilbert-Schmidt class.

In the cases when the spectrum of the given differential operator is dicscrete and the spectrum of the operator consists of infinitely many eigen-values, one of the main problems is to study asymptotic distribution of eigen-values. Discreteness of the spectrum of a higher order elliptic operator in $n-$dimensional Euclidean space and asymptotic distribution of its eigen-values was studied by A.G.Kostyuchenko.

In infinite domains such type problems were considered by V.A.Mikhaylets, A.N.Kojevnikov, K.Kh.Boymatov, Sh.G.Bayimov, H.I.Aslanov and others.

M.Bayramoglu H.I.Aslanov, A.A.Abudov, B.I.Aliyev, G.I.Gasymov, K.H.Badalova and others have studied the Green function of higher order operator-differential equations in Hilbert space and distribution of their eigen values.

The represented dissertation work consists of scientific studies conducted in this direction.

In the dissertation work, the Green function of higher order normal operator coefficiens differential equations given in Hilbert space in semi-axis was constructed, its asymptotic properties were studied, its derivatives were researched, regular estimation was obtained, the discreteness of the spectrum was proved. In the semi-axis, the structure of the spectrum of second order operator-differential equations was studied the discreteness of the negative spectrum was proved, the number of negative eigen-values was estimated, asymptotic distribution formula of eigen values was obtained. It was proved that in the finite segment the resolvent of the $2n-$order operator-differential equation is a Hilbert-Shmidt type operator. The discretness of the spectrum of $2m$-th order elliptic operator-differential equations with special derivatives given in the space R^n was proved, asymptotic distribution formula of eigen values was obtained.

Object and subject of research. Study of the Green function of differential equations with operator coefficients, study of the structure of the negative spectrum of second-order operators, study of the resolution of high-order operator-differential equations and the asymptotic distribution of eigenvalues.

Goal and duties of the research is to study the Green function of higher order normal operator coefficient differential equations on a semi-axis, to study the derivatives of the Green function, to obtain regular estimation, to prove the discreteness of the spectrum on the semi-axis, to study the structure of the negative spectrum of second order operator-differential equations on the semi-axis, to prove its discreteness, to estimate the number of eigen-values, to obtain the asymptotic distribution formula, to prove that the resolvent of a higher order operator-differential operator on a finite domain is a Hilbert-Shmidt type operator.

Research methods. In the dissertation work, the methods of spectral theory of self-adjoint and normal operators in Hilbert space, theory of differential and integral equations, theory of operator-valued functions in Hilbert space and variation methods of functional analysis were used.

The main thesis to be defended.

1. To construct the Green function of higher order normal operator coefficient differential equations on a semi-axis and to study their main properties.
2. To research the derivatives of the Green function, to obtain their regular estimation and to prove the discreteness of the spectrum. .
3. To study the construction of negative spectrum of second order operator-differential equations, on a semi-axis, to prove the discreteness of the negative spectrum.
4. To estimate the number of negative eigen values and to obtain asymptotic distribution formula of negative eigen values.
5. To prove that the resolvent of a higher order operator equation in a finite domain is a Hilbert-Schmidt type operator.
6. To prove asymptotic distribution formula of eigen values of $2m$-th order elliptic type operator-differential equations with special

derivatives in the Euclidean space R^n.

Scientific novelty of the study. In the dissertation work the following new scientific results were obtained:

– The Green function of higher order normal operator coefficient differential equations was structured on a semi-axis and its main properties were studied;

– The derivatives of the Green function were studied, regular estimation of the Green function was obtained and the discreteness of the spectrum was proved;

– The structure of second order operator-differential equations was studied on semi-axis and the discreteness of the negative spectrum was proved;

– The estimation of the number of negative eigen-values was obtained and asymptotic distribution formula of eigen-values was proved;

– It was proved that on a finite segment the resolvent of higher order operator-differential equations is a Hilbert-Schmidt type operator;

– The asymptotic distribution formula of eigen-values of $2m$-th order elliptic type operator-differential equations with special derivatives in R^n Euclidean space was proved.

Theoretical and practical value of the study.

The results scientific obtained in the work are of theoretical character. The obtained results can be used when studying spectral properties of differential equations and also in studies in the field of quantum mechanics.

Approbation and application. The results of the work were reported in scientific seminars held at IMM of ANAS at the depatment of “Functional analysis” (prof. H.İ.Aslanov), “Differential equations” (prof. A.B.Aliyev), at the scientific seminars of “Mathematical analysis and function theory” (prof. N.T.Kurbanov) and “Differential equations and optimization” (prof. F.G.Feyziyev) chairs of Sumgayit State University and at the XX Republican scientific conference of doctoral students and young researches held in Azerbaijan State Academy of Oil and Industry (24-25 may, 2016, Baku), in the International Workshop of Non-harmonic analysis and differential operators (25-27 may 2016, Baku), at the Republican scientific conference "Functional analysius and its applications" dedicated to 100 years of honored scientist prof. A.Sh.Habibzade (Baku-2016), at the republican scientific conference "Applied and actual problems of theoretical and applied mathematics" dedicated to 100 years of acad. Rasul Majidov and held together with ANAS, BSU and ASPU (28-29 ooctorer, 2016-cı il, Sheki), in the international conference "Applied problems of mathematics" held by Sumgayit State University together with Institute of Information Technology of ANAS (25-27 may – 2017, Sumgayit), in the republican scientific conference "Actual problems of mathemtics and mechanics" dedicated to 100 years of prof. Goshgar Ahmedov (02-03 november 2017, Baku).

Author's personal contribution. The obtained results and suggestions belong to the author.

Author's publications. The main results of the work were published in 14 papers the list of which is at the end of the author's thesis.

The organization where the dissertation work was done. The dissertation work was done in the chair of "Mathematical analysis and functional theory" of Sumgayit State University.

Total volume of the work. The work consists of introduction - 50000 signs, title page-591 signs, content -2191 signs, 3 chapters-164000 signs (I chapter -72000, II chapter - 54000, III chapter -38000), results-1695 signs and list of references with 121 titles. The total volume of the work consists of 218477 signs.

THE CONTENT OF THE DISSERTATION

The dissertation work consists of introduction and three chapters. In the introduction brief review of scientific works related to the topic of the dissertation work is given and the main results are commented.

Chapter I of the work was devoted to the study of higher order normal operator coefficient differential equations on a semi-axis. The Green function of the equation was constructed, its main properties were studied, regular estimates of the Green function for rather large values of the spectral parameter were obtained. Using these estimates, it was proved that the Green function is a Hilbert-Schmidt-type kernel. Hence it is obtained that the operator under consideration has a discrete spectrum

Assume that H is a seperable Hilbert space. In the space $H_1 = L_2[[0,\infty);H]$ we consider the operator L determined by the differential expression

$$l(y) = (-1)^n y^{(2n)} + \sum_{j=2}^{2n} Q_i(x) y^{(2n-j)} \qquad (1)$$

and the boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = \dots = y^{(l_n)}(0) = 0. \qquad (2)$$

Here $0 \le l_1 < l_2 < \dots < l_n \le 2n-1$.

To study the problem we will assume that the coefficients $Q_j(x)$ $(j = 2,3,\dots,2n)$ of the given differential equation satisfy the following conditions:

1°. For all $x \in [0,\infty)$ in the space H there exists such an everywhere dense $D\{Q(x)\} = D(Q)$ (here we denote $Q(x) = Q_{2n}(x)$) that the operator $Q(x)$ is a normal operator in this domain.

2°. Assume that the operator $Q^{-1}(x)$ for any $x \in [0,\infty)$ is a completely continuous operator. Thus, the eigen values of the operator $Q(x)$ are located on a complex plane outside the domain $\Omega = \{\lambda : |arq\lambda - \pi| < \varepsilon_0, 0 < \varepsilon_0 < \pi\}$.

Let $\alpha_1(x), \alpha_2(x), \dots, \alpha_n(x), \dots$ be eigen values of the operator $Q(x)$ and they are lined up in ascending direction of their module: $|\alpha_1(x)| \le |\alpha_2(x)| \le \dots \le |\alpha_n(x)| \le \dots$ It is assumed that the series,

$\sum_{j=1}^{\infty}\left|\alpha_j(x)\right|^{\frac{1-4n}{2n}}$ is a convergant series for all $x\in[0,\infty)$ and its sum is $F(x)\in L_1[0,\infty)$.

3°. For any $x\in[0,\infty)$ and $|x-\xi|\le 1$ the conditions

$$\left\|[Q(\xi)-Q(x)]Q^{-a}(x)\right\|_H\le A|x-\xi|,$$

$$\left\|Q^{-\frac{1}{2n}}(x)Q^{\frac{1}{2n}}(\xi)\right\|_H<c_1,\left\|Q^{-\frac{1}{2n}}(\xi)Q^{\frac{1}{2n}}(x)\right\|_H<c_2$$

are satisfied. Here A,c_1,c_2 are positive constants and $0<a<\dfrac{2n+1}{2n}$.

4°. For all $x\in[0,\infty)$ and $|x-\xi|>1$ the following inequality is satisfied

$$\left\|Q(\xi)\exp\left(-\frac{Jm\omega_1}{2}|x-\xi|Q^{\frac{1}{2n}}(x)\right)\right\|_H<B, B>0 \text{ is a constant.}$$

$Jm\omega_1=\min\limits_i\{Jm\omega_i>0,\omega_i^{2n}=-1\}$.

5°. For all $x\in[0,\infty)$ the inequality

$$\left\|Q_j(x)Q^{\frac{1-j}{2n}+\varepsilon}(x)\right\|_H<c_3, c_3>0, j=2,3,...,2n-1$$

is satisfied.

The Green function of the operator L is constructed in three stages. In the first stage the Green function of the operator L_1 determined by a differential expression

$$l_1(y)=(-1)^n y^{(2n)}+Q(\xi)y+\mu y \qquad (3)$$

with the coefficient frozen at the point "ξ" and boundary conditions (2) is constructed.

At the second stage the Green function of the operator L_0 determined by the differential expression

$$l_0(y)=(-1)^n y^{(2n)}+Q(x)y+\mu y \qquad (4)$$

With a coeffcideitn dependent on $x-$ and boundary conditions (2) is constructed and some of its properties are studied.

At the third stage, the Green function of problem (1)-(2) is constructed and its properties are studied. Using the estimation of the Green function it is proved it is a Hilbert-Schmidt type kernel.

In section 1.2. the Green function of the operator L_1 is structured: The Green function $G_1(x,\eta,\xi,\mu)$ is sought in the form of

$$G_1(x,\eta,\xi,\mu)=g(x,\eta,\xi,\mu)+V(x,\eta,\xi,\mu) \qquad (5)$$

Here the function $g(x,\eta,\xi,\mu)$ is a Green function of the equation $l_1(y)=0$ on the real axis:

$$g(x,\eta,\xi,\mu)=\frac{[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}}{2n\cdot i}\cdot\sum_{\alpha=1}^{n}\omega_\alpha e^{i\omega_\alpha[Q(\xi)+\mu E]^{\frac{1}{2n}}|x-\eta|} \qquad (6)$$

The function $V(x,\eta,\xi,\mu)$ is a bounded solution of the problem

$$\begin{cases} l_1(y)=0 \\ V^{(l_j)}_{(x,\eta,\xi,\mu)}\Big|_{x=o}=-g^{(l_j)}_{(x,\eta,\xi,\mu)}\Big|_{x=0} \end{cases}, j=1,2,...,n \qquad (7)$$

provided $x\to\infty$.

It is shown that

$$V(x,\eta,\xi,\mu)=\frac{[Q(\xi)+\mu E]^{\frac{1-2n}{2n}}}{2ni}\sum_{\alpha=1}^{n}\omega_\alpha e^{i\omega_\alpha[Q(\xi)+\mu E]^{\frac{1}{2n}}(x+\eta)} \qquad (8)$$

for the Green function $G_1(x,\eta,\xi,\mu)$ the following expression was obtained:

$$G_1(x,\eta,\xi,\mu)=g(x,\eta,\xi,\mu)[E-r(x,\eta,\xi,\mu)] \qquad (9)$$

Here $\|r(x,\eta,\xi,\mu)\|_H=o(1)$, as $\mu\to\infty$.

In section 1.3. using the Levi method, for the Green function of the operator L_0 we obtain a Fredholm type integral equation:

$$G_0(x,\eta,\mu)=G_1(x,\eta,\mu)-\int_0^\infty G_1(x,\xi,\mu)[Q(\xi)-Q(x)]G_0(x,\eta,\mu)d\xi \qquad (10)$$

The solution of these integral equation is studied in Banach spaces consisting of operator-valued functions $X_1, X_2, X_3^{(p)}, X_2^{(s)}, X_5 (p \geq 1, s \leq 0)$ introduced by V.M.Levitan.

To this end, the following operator is determined in the spaces X_1 (or X_2):

$$TA(x,\eta) = \int_0^\infty G_1(x,\xi,\mu)[Q(\xi) - Q(x)]A(\xi,\eta)d\xi \tag{11}$$

Here $\|\alpha(x,\eta,\mu)\| = o(1), \mu \to \infty$.

The following theorem is proved.

Theorem 1.3.1. If the operator-function $Q(x)$ satisfies conditions $1^\circ - 4^\circ$, then for all rather large values of the parameter $\mu > 0$ the operator T is a compressive operator in the spaces X_1 (or X_2) .

From this theorem we get that the integral equation (10) is a unigue solution in the spaces X_1, X_2 and the asymptotics equality

$$G_0(x,\eta,\mu) = G_1(x,\eta,\mu)[E + \alpha(x,\eta,\mu)] \tag{12}$$

In section 1.4. we study the derivatives of the Green function $G_0(x,\eta,\mu)$. It was proved that the derivatives $\frac{\partial^k G_0(x,\eta,\mu)}{\partial \eta^k}$ $(k = 1,2,...,2n-2)$ are continuous operator-functions with respect to the variables x,η in the strong sense. For the values $\eta \neq x$ the derivative $\frac{\partial^{2n-1} G_0(x,\eta,\mu)}{\partial \eta^{2n-1}}$ is a continuous function, but for the values $\eta = x$ it has first kind discontinuity and at this point it has the "jump'

$$\frac{\partial^{2n-1} G_0(x,x+0,\mu)}{\partial \eta^{2n-1}} - \frac{\partial^{2-1} G_0(x,x-0,\mu)}{\partial \eta^{2n-1}} = (-1)^n E .$$

It was shown that the function $G_0(x,\eta,\mu)$ satisfies the equation

$$(-1)^n G_{0\eta}^{(2n)}(x,\eta,\mu) + G_0(x,\eta,\mu)[Q(\eta) + \mu E] = 0$$

and the boundary conditions

$$\left.\frac{\partial^{l_1} G_0(x,\eta,\mu)}{\partial \eta^{l_1}}\right|_{\eta=0} = \left.\frac{\partial^{l_2} G_0(x,\eta,\mu)}{\partial \eta^{l_2}}\right|_{\eta=0} = \dots = \left.\frac{\partial^{l_n} G_0(x,\eta,\mu)}{\partial \eta^{l_n}}\right|_{\eta=0} = 0 \cdot$$

In section 1.5. the Green function of problem (1)-(2) was constructed and provided $\mu \to \infty$ its regular estimation with respect to the variables (x,η) was obtained.

The Green function $G(x,\eta,\mu)$ is sought in the form

$$G(x,\eta,\mu) = G_0(x,\eta,\mu) - \int_0^\infty G_0(x,\xi,\mu)\rho(\xi,\eta)d\eta \qquad (13)$$

For the function $\rho(x,\eta)$ included into the equality (13) we get a Fredholm type equation

$$\rho(x,\eta) = F(x,\eta,\mu) - \int_0^\infty F(x,\eta,\mu)\rho(\xi,\eta)d\xi \qquad (14)$$

Here

$$F(x,\eta,\mu) = -\sum_{j=2}^{2n} Q_j(x)\int_0^\infty \frac{\partial^{2n-j} G_0(x,\xi,\mu)}{\partial \eta^{2n-j}} \qquad (15)$$

Using the equality (15), we get the following estimations:

$$\|F(x,\eta,\mu)\|_H \le c\cdot\mu^{-\varepsilon}\cdot\delta_0^{-\varepsilon}\cdot e^{-r_0 Jm\omega_1\cdot\mu^{\frac{1}{2n}}\cdot\delta_0^{\frac{1}{2n}}|x-\eta|} \qquad (16)$$

$$\sup_{0\le x<\infty}\int_0^\infty \|F(x,\eta,\mu)\|_H^2 d\eta \le \frac{c^1}{\mu^\beta}, \beta > 0 \qquad (17)$$

From inequality (17) we get that $\|F(x,\eta,\mu)\| \in X_3^{(2)}$ and at the some time $\|F(x,\eta,\mu)\|_{X_3^{(2)}} \to 0$.

As a result as $\|\rho(x,y) - F(x,\eta,\mu)\|_{X_3^{(2)}} \to 0$ and at rather large values of the parameter μ we get

$$G(x,\eta,\mu) = G_0(x,\eta,\mu)[E + \sigma(x,\eta,\mu)],\ \ \|\sigma(x,\eta,\mu)\|_H = o(1) \quad (18)$$

From equalitites (10), (13) and (18) as a final result we get

$$G(x,\eta,\mu) = g(x,\eta,\mu)[E + \beta(x,\eta,\mu)],\ \ \|\beta(x,\eta,\mu)\|_H = o(1) \quad (19)$$

From the equality (19) we get

$$\int_0^\infty \int_0^\infty \|G(x,\eta,\mu)\|_H^2 \, dx d\eta < \infty \qquad (20)$$

From the condition (20) we get that the function $G(x,\eta,\mu)$ is a Hilbert Schmidt type ketrnel. Since the kernel $G(x,\eta,\mu)$ is a kernel of the operator $R_\mu = (L+\mu E)^{-1}$ in the space H_1 we get that the operator L has a discrete spectrum.

Chapter II of the dissertation work was devoted to the study of the structure of the negative spectrum of a second order operator-differential equation on a semi-axis, to the proof of discreteness of the negative spectrum, to estimating the number of negative eigen-values and to obtaining asymptotic formula for the number of negative eigen-values.

Assume that H is a separable Hilbert space. In the space $H_1 = L_2[[0,\infty);H]$ we consider an operator L determined by the differential expression

$$l(y) = -(P(x)y')' - Q(x)y \qquad (21)$$

and the boundary condition

$$y'(0) = 0 \qquad (22)$$

Assume that the coefficients, $P(x)$ and $Q(x)$ of the operator L satisfy the following conditions:

1) $P(x)$ is a continuous scalar function with a continuous derivative in the interval $[0,\infty)$ and these exist such constants $c_1 > 0, c_2 > 0$ that $c_1 < P(x) < c_2$.

2) For any $x \in [0,\infty)$ $Q(x)$ is a continuous positive, monotone decreasing operator and $\|Q(x)\|_H$ is a continuous function so that

$$\lim_{x\to\infty} \|Q(x)\|_H = 0$$

In section 2.1. the obtained main result consists of the following theorem.

Theorem 2.1.1. If the coefficients of the differential operator L satisfy the conditions 1),2), then L is a lower bounded operator and negative part of its spectrum is discrete.

Let $\alpha_1(x)$ be an first eigen value of the operator $Q(x)$. Denote by $\alpha_1(x)=\|Q(x)\|_H$. $\psi_1(x)$ an inverse function of the function $\alpha_1(x)$on the interval $(0,\alpha_1(0))$.

Denote by L_1 and L_2 the operators determined in the space $\widetilde{H}_{01}=L_2[[0,\psi_1(\varepsilon)];H]$ by the differential equation (21) and the boundary conditions

$$y(0)=y[\psi_1(\varepsilon)]=0 \tag{23}$$

$$y'(0)=y'[\psi_1(\varepsilon)]=0 \tag{24}$$

the number of eigen values of the operator L,L_1,L_2 less than $(-\varepsilon)-$ by $N(\varepsilon),N_1(\varepsilon)$ and $N_2(\varepsilon)$ respectively.

The following auxiliary lemma is valid:

Lemma 2.2.1. If the coefficients of the operator L satisfy conditions 1),2) then the relations $N(\varepsilon)\geq N_1(\varepsilon)$ and $N(\varepsilon)\leq N_2(\varepsilon)$ are valid.

By L_{01} we denote an operator determined in the space $H_1=L_2[[0,\infty);H]$ by the differential expression

$$l_0(y)=-y''-c_1^{-1}Q(x)y \tag{25}$$

and the boundary conditions

$$y'(0)=0, \tag{26}$$

by L_{02} an operator determined in the space $\widetilde{H}_{02}=L_2[[\psi_1(\varepsilon),\infty);H]$ by differential expression (24) abd the boundary condition $y'[\psi_1(\varepsilon)]=0$. By $o=x_0<x_1<x_2<...<x_m=\psi_1(\varepsilon)$ we denote divison $\delta=\frac{\psi_1(\varepsilon)}{[\psi_1^k(\varepsilon)]+1}$ that is the length of the segment $[0,\psi_1(\varepsilon)]$.

In the space $L_2[[x_{i-1},x_i];H]$ we introduce the following operators:

By $L_{i(1)}$ we denote an operator determined by the differential expression

$$l_{i(1)}(y) = -p(x_i)y''(x) - Q(x_i)y \qquad (27)$$

and boundary conditions $y(x_{i-1}) = y(x_i) = 0$ by $L_{i(1)}$ the operator determined by the differential expression

$$l_{i(2)}(y) = -p(x_{i-1})y''(x) - Q(x_{i-1})y \qquad (28)$$

by $L_{i(2)}$ and the boundary conditions $y'(x_{i-1}) = y'(x_i) = 0$.

The following lemma is valid:

Lemma 2.2.2. Assume that conditions, 1),2) are satisfied. Then the inequlities $L_{(1)i} < L_{i(1)}$ and $L_{(2)i} < L_{i(2)}$ are valid.

In section 2.2. the following main theorems are proved.

Theorem 2.2.2. If the coefficients of the operator L satisfy conditions 1),2) sense for the number of the negative values of the operator L the following inequalities is valid:

$$N(\varepsilon) > \frac{1}{\pi}\sum_{i=1}^{l_\varepsilon}\int_0^{\psi_i(\varepsilon)}\sqrt{\frac{\alpha_i(x)-\varepsilon}{p(x)}}\,dx - c\cdot l_\varepsilon\int_0^{\delta}\sqrt{\alpha_1(x)}\,dx - $$
$$-c\cdot l_\varepsilon\psi_1^k(\varepsilon) \qquad (29)$$

Here $l_\varepsilon = \sum\limits_{\alpha_1(0)>\varepsilon} 1.$

Theorem 2.2.5. If the coefficients of the operator L satisfy conditions 1),2) then for rather small values of $\varepsilon > 0$ – the following estimation is valid:

$$N(\varepsilon) < \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_i(x)-\varepsilon}{p(x)}}\,dx + const\cdot l_\varepsilon\cdot\int_0^{\delta}\sqrt{\alpha_1(x)}\,dx + const\cdot l_\varepsilon\cdot\psi_1^k(\varepsilon) \qquad (30)$$

In section 2.3. the asymptotic distribution formula of negative eigen values of the operator L determined by differential expression (20) and boundary conditions (21) was proved.

It is assumed that the coefficients $p(x)$ and $Q(x)$ satisfy conditions 1),2) and further more the function $\alpha_1(x)$ being the first eigen values of the operator $Q(x)$ satisfies the following conditions:

3) For arbitrary number $\eta > 0$

$$\lim_{x\to\infty} \alpha_1(x)x^{k_0-\eta} = \lim_{x\to\infty}\left[\alpha_1(x)x^{k_0+\eta}\right]^{-1} = 0,$$

here $k_0 \in (0,2)$ is any number.

4) The series $\sum_{i=1}^{\infty}[\alpha_i(0)]^m$ is a convergent series, $0 < m < \frac{(2-k_0)^2}{8k_0 - 2k_0^2}$.

The following theorem was proved:

Theorem 2.3.1. Assume that the coefficients of the operator L satisfy conditons 1)-4). Then provided $\varepsilon \to 0$ the following asymptotic formula is valid:

$$N(\varepsilon) = \frac{1}{\pi}[1 + O(\varepsilon^{-t_0})]\sum_{j=1}^{l_\varepsilon}\int_{\alpha_j(x)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx \qquad (31)$$

Here t_0 is some positive number.

Chapter III of the work devoted to studying the resolvent of higher order operator-differential equations in finite segment in Hilbert space and obtaining asymptotic distribution formula of eigen values of a class of partial elliptic type operator differential equations.

In section 3.1. in a finite segment a resolvent higher order operator-differential equations was studied and it was proved that the resolvent is an Hilbert-Schmidt type integral operator.

In the space $L_2[[0,\pi];H]$ we consider the operator L determined by the differential expression

$$l(y) = (-1)^n\left(P(x)y^{(n)}\right)^{(n)} + Q(x)y \qquad (32)$$

and the boundary conditions

$$\begin{cases} y(0) = y'(0) = \dots = y^{(n-1)}(0) = 0 \\ y(\pi) = y'(\pi) = \dots = y^{(n-1)}(\pi) = 0 \end{cases}. \qquad (33)$$

Assume that the coefficitns $P(x)$ and $Q(x)$ satisfy the following conditions:

1) For any $x \in [0, \pi]$ $P(x)$ is $n-$times regularly differentiable operator-function and for any $h \in H$ the condition
$m(h,h)_H \le (P(x)h,h)_H \le M(h,h)_H, m, M > 0$
is satisfied.

2) For any $x \in [0, \pi]$ there exists such a general set $D\{Q(x)\} = D(Q)$ everywhere dense in the space H that the operator $Q(x)$ is a self-adjoint, lower-bounded operator defined in this set, i.e. for any $f \in D(Q)$ $(Q(x)f, f)_H > (f, f)_H$ is satisfied.

3) There exist such constant numbers $A > 0,\ 0 < a < \dfrac{2n+1}{2n}$ that for arbitrary $x \in [0, \pi]$ $\left\| [Q(x) - Q(\xi)]Q^{-a}(x) \right\|_H < A|x - \xi|$ is satisfied for $|x - \xi| \le 1$.

4) For arbitrary $x \in [0, \pi]$ and $|x - \xi| > 1$

$$\left\| K(\xi) \exp\left(-\frac{Jm\varepsilon_1}{2} |x - \xi| \omega \right) \right\|_H < B,$$

here $K(\xi) = P(\xi)^{\frac{1}{2}} Q(\xi) P^{-\frac{1}{2}}(\xi),\ \omega = \left[K(\xi) + \mu P^{-1}(\xi)\right]^{\frac{1}{2n}},\ \mu > 0,$ $Jm\varepsilon_1 = \min\limits_i \{Jm\varepsilon_i > 0, \varepsilon_i^{2n} = -1\}.$

5) For arbitrary $x, \xi \in [0, \pi]$ the condition

$$\left\| Q(x) P^{\pm\frac{1}{2}}(x) Q^{-1}(x) \right\|_H < c, \left\| Q(\xi) P^{-\frac{1}{2}}(x) P^{\frac{1}{2}}(\xi) Q^{-1}(\xi) \right\|_H < c.$$

is satisfied.

6) The operator $Q^{-1}(x)$ is a completely continuous operator. (This time the operator $K^{-1}(x)$ is also a completely continuous operator). By $\beta_1(x) \le \beta_2(x) \le ... \le \beta_n(x) \le ...$ we denote eigen values of the operator $K(x)$ lined up in an ascending direction so these eigen values. These eigen-values are measurable functions. The

series $\sum_{k=1}^{\infty} \beta_k^{\frac{1-4n}{2n}}(x)$ is convergent for any $x \in [0, \pi]$ and its sum is $F(x) \in L_1[0, \pi]$. When these conditions are satisfied, it was proved that the spectrum of the operator L is discrete. Denote by $\lambda_1, \lambda_2, ..., \lambda_n, ..$ the eigen values of the operators L. L is an unbounded operator and provided $n \to \infty$ $\lambda_n \to \infty$.

The following main theorem was proved.

Theorem 3.1.1. If the coefficients of the operator L satisfy conditions 1)-6), then the series $\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2}$ is convergent i.e. the operator L is a Hilbert-Schmidt type operator.

In section 3.2. we study the discreteness of the spectrum of higher order elliptic type partial operator-differential equations and asymptotic distribution of eigen-values.

Assume that H is a separable Hilbert space. In the space $H_1 = L_2(R^n; H)$ we consider the operator L determined by the differential expression

$$l(y) = (-1)^m \sum_{k_1+k_2+...+k_n=2m} A^{k_1 k_2 ... k_n}(x) \frac{\partial^{2m} u}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n}} + Q(x)u, \qquad (34)$$

It is assumed that the principle part of differential expression (34) satisfies the regular ellipticity condition, i.e. there exist such constants c_1, c_2 that

$$c_1 |s|^2 \le \sum_{k_1+k_2+...+k_n=2m} A^{k_1 k_2 ... k_n}(x) s_1^{k_1} s_2^{k2} ... s_n^{kn} \le c_2 |s|^{2m}$$

here $s = (s_1, s_2, ..., s_n) \in R^n$.

Assume that the coefficients of expression (34) satisfy the following conditions:

1. $A^{k_1 k_2 ... k_n}(x)$ is a bounded function with real values in the space R^n and satisfies the Holder condition:

$$\left|A^{k_1k_2...k_n}(x)-A^{k_1k_2...k_n}(\xi)\right|\le K|x-\xi|^{\gamma}, |x-\xi|<1, 0<\gamma<1.$$

2. $Q(x)$ is a self-adjpoint, lower regularly bounded operator and $Q^{-1}(x)\in\sigma_{\infty}$.

3. For some $l>0$ $Q^{-l}(x)\in\sigma_1$, $x\in R^n$ and $\int_{R^n}\left\|Q^{-l}(x)\right\|_1 dx<\infty$.

4. There exists such a function $f(x)$ that for $|x-\xi|\le 1$

$\left\|e^{-ctQ(\xi)}\right\|_1\le\left\|e^{-f(c)tQ(\xi)}\right\|_1$ is satisfied.

Here $c>0, t>0$ $\left(f(c)>0\right)$ is any number.

5. For any number $M>0$

$$\int_{R^n} tre^{-MtQ(x)}dx=\overline{\overline{O}}(1)\int_{R^n} tre^{-tQ(x)}dx$$

6. By $\alpha_1(x)\le\alpha_2(x)\le...\le\alpha_n(x)\le...$ we denote eigen values of the operator $Q(x)$ in the space H.

Determined the function.

$$\Phi(x)=\int_{R^n}e^{-L_0(x,is)}ds,\ \rho(\lambda)=\sum_{i=1}^{\infty}\int_{\alpha_1(x)<\lambda}\Phi(x)[\lambda-\alpha_i(x)]^{\frac{n}{2m}}dx.$$

Assume that for rather large values of $\lambda>0$ the condition $\lambda\rho'(\lambda)<\alpha_0\rho(\lambda)$ is satisfied.

The following theorem was proved.

Theorem 3.2.1. If the coeffcients of differential expression (34) satisfy conditions 1.-6, then the spectrum of the operator L is discrete and for the function $N(\lambda)=\sum_{\lambda_n<\lambda}1$ showing the number of eigen values less than the number λ the following asymptotic formula is valid as $\lambda\to\infty$:

$$N(\lambda)\sim\frac{1}{(2\pi)^n\Gamma\left(\frac{n}{2m}+1\right)}\sum_{i=1}^{\infty}\int_{\alpha_1(x)<\lambda}\Phi(x)[\lambda-\alpha_i(x)]^{\frac{n}{2m}}dx. \quad (35)$$

CONCLUSION

In the dissertation work the Green function of higher order operator coefficiens differential equations given on a semi-axis in Hilbert space was constructed, its asymptotic properties are studied the derivatives were researche, regular estimation is conducted and discreteness of its spectrum was proved.

The structure of the spectrum of second order operator-differential equations on a semi-axis was studied, the discreteness of the negative spectrum was proved the number of negative eigen-values was estimated asymptotic distributions formula of eigen values was obtained. It was proved, that in a finite segment the resolvent of $2n-$order operator-differential equation is a Hilbert-Schmidt type operator. The discretness of the spectrum of $2m$ order elliptic operator-differential equations given in the space R^n was proved, asymptotic distribution formula of eigen values was obtained.

The following new results were obtained:

– The Green function of higher order normal operator coefficient differential equations was constructed and its main properties were studied;

– The derivatives of the Green function were studied, regular estimation of the Green function was obtained and the discreteness of the spectrum was proved;

– The structure of the spectrum of second order operator-differential equations in a semi-axis was studied, the discreteness of the negative spectrum was proved;

– The estimations of the number of negative eigen-members were obtained, asymptotic distribution formula of eigen-values was proved;

– It was proved that the resolvent of higher order operator-differential equations on a finite segment is a Hilbert-Schmidt type operator;

– In the Euclidean space R^n the asymptotic distribution formula of eigen-values of $2m$-th order elliptic type operator-differential equations with special derivatives was proved.

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1. Гадирли, Н.А. О резольвенте операторного уравнения высокого порядка с номальным операторным коэффициентом на полуоси // Doktorantların və Gənc Tədqiqatçıların XX Respublika elmi konfransının materialları, I cild, -Bakı:-2016, -s. 24-25.
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3. Гадирли, Н.А. Распределение собственных значений эллиптических операторно-дифференциальных уравнений высокого порядка в Гильбертовом пространстве //"Funksional analiz və onun tətbiqləri" əməkdar elm xadimi, professor Əmir Şamil oğlu Həbibzadənin anadan olmasının 100-cü ildönümünə həsr olunmuş Respublika elmi konfransın materialları, -Bakı: -2016, -s. 121-123.
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