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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

OPTIMAL CONTROL PROBLEMS FOR SOME THIRD ORDER PARTIAL EQUATIONS

Specialty: 1214.01 – Dynamical systems and optimal control

Field of science: Mathematics

Applicant: **Mehriban Yagub kizi Yagubova**

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The work was performed at the Scientific Research Institute of Applied Mathematics of Baku State University.

Scientific supervisor:

academician of NASA, doctor of phys-math.sc. professor
Fikrat Ahmedali oglu Aliyev

Scientific consultant:

doctor of phys-math.sc. professor
Aleksandr Petrovic Afanasyev

Official opponents:

doctor of phys-math.sc. professor
Misraddin Allahverdi oglu Sadigov

doctor of phys-math.sc. professor
Yagub Amiyar oglu Sharifov

doctor of sciences in math. Assoc.
Ilgar Gurbat oglu Mammadov

Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

Chairman of the Dissertation council:

corr.-member of ANAS, doctor of phys.-math. sc., prof.
_____________ **Misir Jumail oglu Mardanov**

Scientific secretary of the Dissertation council:

cand. of phys.-math.sc.
_____________ **Abdurrahim Farman oglu Guliyev**

Chairman of the scientific seminar:

doctor of physical and mathematical sciences, professor
_____________ **Hamlet Farman oglu Quliyev**

GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic.

Despite the fairly intensive development of theory of optimal control in processes described by partial equations, scientific interest in such problems does not diminish, but on the contrary increases. The main reason for this is that, in connection with application to the solution of new applied problems, it is necessary to consider equations of higher orders. And this in its turn leads to formulation of new mathematical problems and creation of new schemes for their research. As an example we can show the problems related to third and fourth order partial equations. It should be noted that optimal control problems related to fourth order equations were studied in the papers of V.Komkov, J.L.Lions, A.I.Egorov, T.K.Sirazetdinov, G.F.Guliyev, A.A.Mekhtiyev, V.B.Nazarova. In all these papers necessary conditions for optimality, sufficient conditions for optimality were obtained, theorems on the existence of optimal control were proved and optimal control was constructed as the solution of a synthesis problem or in the form of a series.

For third order equations or the so called variable type equations some results of M.A.Yagubov, R.B.Huseynov are known, where necessary conditions were obtained, an optimal control was obtained in the form of a series in the case of one spatial variable.

This dissertation work also was devoted to the study of various problems of optimal control in the processes described by variable type equations in the case of many spatial variables. The results obtained in the dissertation work can be used for solving applied problems. Therefore the topic of the work is actual.

Object and subject of research. Initial-boundary value problem and optimal control problems for third-order partial differential equations

The goal of the study. Deriving various conditions for optimality and sufficient conditions for optimality, studying controllability problems, constructing optimal distributed and start controls in the case of a linear equation with two spatial variables with a quadratic quality criterion.

General research methodology. In the dissertation work methods of functional analysis, Fourier method, methods of mathematical theory of optimal control and theory of double functional series are used.

Main thesis to be definded.

- derivation of the formula for the gradient of the functional with distributed control on the right side of the third order equation,
- derivation of necessary optimality conditions in the form of the maximum principle,
- derivation of integral optimality conditions,
- derivation of sufficient optimality conditions in the case of a convex functional and a convex set of control values,
- proof of the differentiability of the quadratic functional in the case of a linear equation in the presence of distributed and start controls,
- construction of optimal control in the form of a double series and proof of its convergence,
- reduction of the controllability problem with minimum energy in the presence of distributed and starting controls is reduced to a conditional extremum problem, obtaining a solution in the form of a double series and proving its convergence,
- construction in the form of a double series of the solution of the problem with minimum energy in the presence of distributed and two start controls,
- show the applicability of the problem of moments to the solution of the problem of stabilization in the case when the solution of the mixed problem vanishes at a finite moment of time,
- reduction of the stabilization problem to a conditional extremum problem and construction of the solution of this problem in the form of a convergent double series.

Scientific novelty.

The following main results were obtained in the dissertation:

1. For processes described by a third-order nonlinear equation with many independent variables, the necessary optimality conditions are derived in the form of the maximum principle and integral optimality conditions.

2. A formula is obtained for the gradient of the functional in the presence of distributed control.

3. The sufficiency of the conditions is proved under the assumption that the functional is convex with respect to the control variables and the set of values of these variables is convex.

4. In the presence of distributed and start controls, the differentiability of the quadratic functional is substantiated when the process is described by a third-order linear equation with a control function and in the initial condition.

5. The possibility of using the method of separation of variables in the case of two spatial variables is shown and are constructed in the form of double series, both for the solution of a mixed problem and for optimal control. The convergence of these series is proved.

6. A solution of the controllability problem in the form of double series in the presence of distributed and start controls is constructed, reducing it to a problem for a conditional extremum of a function of two independent variables.

7. Solutions in the form of double series of the minimum energy problem in the presence of distributed and two start controls are constructed, and the convergence of these series is proved.

8. The applicability of the problem of moments to the stabilization problem in one case is proved, this problem is reduced to a conditional extremum problem, a solution is constructed in the form of a descending double series.

Theoretical and practical value.

The equations considered in the dissertation work describe gas dynamics problems, turbulence, combustion and other processes. But formulation of optimal control problems and the obtained results are of theoretical character. The methods and schemes given in the work can be used for studying optimal control problems in the processes described by higher order equations.

Approbation of the work. The results of the work obtained at various times were reported in the seminars of the Institute of Applied Mathematics of Baku State University (acad. F.A.Aliyev), in the conferences devoted to 85-th anniversary of corr.-member of

ANAS Ya.Y.Mamedov (Baku, December 10, 2015), to the 100-th anniversary of the honored scientist, prof. A.Sh.Habibzadeh (Baku, June 22-23, 2016), to the 100-th anniversary of acad. of ANAS M.L.Rasulov (Baku, October 28-29, 2016), in the III Republican conference "Applied problems of mathematics and new information technologies" (Sumgayit, December 15-16, 2016), in the Republican scientific conference devoted to the 100-th anniversary of corr.-member of ANAS K.T.Akhmedov (Baku, November 02-03, 2017), in the VI International conference "Control and optimization using in industry" (Baku July 11-13, 2018), in the international scientific conference (UFA, March 10-14, 2020).

The author's personal contribution is in formulation of the research goal. Are the obtained results belong to the author.

Authors publication. Publication in the editions recommended by Higher Certification Commision under the President of the Republic of Azerbaijan – 6. In addition to them including in the journal with the impact factor Web of Science - 1, in the journal with the impact factor Scopus - 1, conference materials – 2, anstracts of papers – 5.

The institution where the dissertation work was performed. The work was performed at the Baku State University.

Structure and volume of the dissertation (in signs indicating the volume of each structural subdivision separately). Total volume of the dissertation work–185456 signs (title page – 462 signs, contents – 2099 signs, introduction – 36137 signs, chapter I – 40000 signs, chapter II – 58000 signs, chapter III 48000 signs, conclusion – 758 signs). The list of references consists of 56 names.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation work consists of introduction, three chapters, conclusion and list of references. Rationale of the topic is justified and brief content of the obtained results are given in the introduction.

Chapter I consists of two sections.

In 1.1 we consider a problem of minimum of the functional

$$I(u) = \int_Q \Phi(x,t,z,z_x,z_t,u)dxdt \qquad (1)$$

on the solutions of the problem

$$\beta z_{tt} + z_t - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial t} F_i(x,t,z_x) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(x,t,z_x) = f_1(x,t,u), \qquad (2)$$

$$z(x,0) = \varphi_1(x), \;\; z_t(x,0) = \varphi_2(x), \;\; z\big|_\Gamma = 0, \qquad (3)$$

where $Q = \Omega \times (0,T)$, $\Omega \subset R^n$ is a domain with the boundary $\partial\Omega$ of the class $C^{2+\alpha}$, $\Gamma = \Omega \times (0,T)$, $z_x = \left(\frac{\partial z}{\partial x_1}, ..., \frac{\partial z}{\partial x_n} \right)$, T is a fixed moment of time, β is a positive constant, $u = (u_1,...,u_r)$ is a control vector and, the vector functions $u = u(x,t)$, measurable on Q and with values from the bounded set $U \in R^r$ are taken as admissible controls.

Throughout this chapter it is assumed that the following conditions are fulfilled:

1. $F_i(x,t,z_x)$, $i = 1,2,...,n$ are twice continuously differentiable functions with respect to all the arguments and

a) $K_0 \sum_{i=1}^{n} \left(1 + |\xi|^{p-2}\right)\xi_i^2 \le \sum_{i=1}^{n} F_i(x,t,\xi)\xi_i \le K \sum_{i=1}^{n} \left(1 + |\xi|^{p-2}\right)\xi_i^2$,

b) $\sum_{i,j=1}^{n} \frac{\partial F_i(x,t,\xi)}{\partial \xi_i} \eta_i \eta_j \ge K_0 \sum_{i=1}^{n} \left(1 + |\xi|^{p-2}\right)\eta_i^2$,

c) $\left| \frac{\partial F_i(x,t,\xi)}{\partial \xi_j} \right| \ge K\left(1 + |\xi|^{p-2}\right)$,

d) $\left| \frac{\partial F_i(x,t,\xi)}{\partial t} \right| + \left| \frac{\partial F_i(x,t,\xi)}{\partial x_j} \right| + |F_i(x,t,\xi)| \ge K\left(1 + |\xi|^{p-1}\right)$, $i, j = \overline{1,n}$;

2. $A_i(x,t,\xi)$ are twice continuously differentiable functions with respect to all the arguments of the function and

a) $|A_i(x,t,\xi)|^2 + \left| \frac{\partial A_i(x,t,\xi)}{\partial x_i} \right|^2 \le C\left(1 + |\xi|^p\right)$,

b) $\left|\dfrac{\partial A_i(x,t,\xi)}{\partial \xi_j}\right|^2 \le C\left(1+|\xi|^{p-2}\right),$

where K_0, K, C are positive constants, $\xi = (\xi_1,...,\xi_n), p \ge 2$.

3. The function $f_1(x,t,u)$ is continuously differentiable with respect to all the arguments.

Subject to this condition it is clear that for each admissible control $u = u(x,t)$ the function $f_1(x,t,u(x,t))$ will be a measurable function, and $f_1(x,t,u(x,t)) \in L_2(Q)$.

4. The functions $\varphi_1(x), \varphi_2(x)$ satisfy the conditions:

$$\varphi_1 \in \overset{0}{W}_{2p}(\Omega),\ \left|\frac{\partial \varphi_1}{\partial x_i}\right|^{p-1} \cdot \frac{\partial \varphi_1}{\partial x_i} \in W_2^1(\Omega),\ \varphi_2 \in L_2(\Omega).$$

Note that subject to these conditions, for each admissible control the initial-boundary value problem (2), (3) has a generalized solution under which we understand the function $z(x,t)$,

$$z \in L_p\left(0,T;\overset{0}{W}{}_p^1(\Omega)\right), u_t \in L_\infty(0,T;L_2(\Omega) \cap L_2(0,T)); \overset{0}{W}{}_p^1(\Omega), B^{1/2}\Delta_x z \in L_2(0,T;L_2(\Omega))$$

satisfying for any $G(x,t) \in L_p\left(0,T;\overset{0}{W}{}_p^1(\Omega)\right)$ the integral identity

$$\int_Q (\beta z_t(x,t) + z(x,t))G(x,t)dxdt + \int_Q \sum_{i=1}^n F_i(x,t,z_t(x,t))G_{x_i}(x,t)dxdt +$$
$$+\int_Q \sum_{i=1}^n G_{x_i}(x,t)\int_0^t A_i(x,\tau)z_x(x,\tau)d\tau dxdt = \int_Q (\beta\varphi_2(x,t) + \varphi_1(x,t))G(x,t)dxdt +$$
$$+\int_Q \sum_{i=1}^n F_i(x,0,\varphi_{1x}(x))G_{x_i}(x,t)dxdt + \int_Q G(x,t)\int_0^t f_1(x,\tau,u(x,\tau))dxdt$$

and the initial condition $z(x,0) = \varphi_1(x)$, where $B(x)$ is a non-negative, smooth finite in Ω function, Δ is a Laplace operator.

For $\Phi(x,t,z,\xi,\eta,u)$ we suppose that

5. $\Phi(x,t,z,\xi,\eta,u)$ is a twice continuously differentiable function with respect to all the arguments and

$$\left|\Phi(x,t,z,\xi,\eta,u)\right| \le a_0 + a_1\left(\ |z|^2 + |\xi|^2 + |\eta|^2 \right),$$

where a_0, a_1 are some positive constants,

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2, \quad |\eta|^2 = \eta_1^2 + \eta_2^2 + \ldots + \eta_n^2 .$$

Under these assumptions, a formula for the increment of the functional is derived, an estimation for its remainder is obtained. Based on these estimations the following theorem is proved.

Theorem 1. Let conditions 1-5 be fulfilled. Then the functional (1) determined on the solutions of initial boundary value problem (2), (3) is Gateaux differentiable in $L_2(Q)$ and

$$gradI(u) = -H_u(x,t,z,z_x,z_t,\psi,u),$$

where $\psi(x,t)$ is the solution of the conjugate problem:

$$\beta\frac{\partial^2\psi}{\partial t^2} - \frac{\partial\psi}{\partial t} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\cdot\left(\frac{\partial^2\psi}{\partial x_i\partial t}\cdot\frac{\partial F_i(x,t,z_x)}{\partial z_{x_j}}\right)+$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\cdot\left(\frac{\partial\psi}{\partial x_i}\cdot\frac{\partial A_i(x,t,z_x)}{\partial z_{x_j}}\right) - \frac{\partial H(x,t,\psi,u)}{\partial z}+$$

$$+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial H(x,t,\psi,u)}{\partial z_{x_j}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial H(x,t,\psi,u)}{\partial z_t}\right) = 0,$$

$$\psi(x,T) = 0, \quad x \in \overline{\Omega},$$

$$\beta\frac{\partial\psi(x,T)}{\partial t} + \frac{\partial H(x,T,\psi,u)}{\partial z_t} +$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\cdot\left(\frac{\partial\psi(x,T)}{\partial x_i}\cdot\frac{\partial F_i(x,T,z_x(x,T)}{\partial z_{x_j}}\right) = 0, \ x \in \Omega,$$

$$\sum_{i=1}^{n}\left(\frac{\partial\psi}{\partial t}\frac{\partial F_i}{\partial z_{x_j}} - \psi\frac{\partial A_i}{\partial z_{x_j}}\right)\cos(\nu,x_i) = 0, \ x \in \partial\Omega, \ t \in (0,T), \ j = 1,2,\ldots,n;$$

$$\psi(x,0) = 0, \quad x \in \partial\Omega, \ j = 1,2,\ldots,n,$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \psi(x,T)}{\partial x_i} \cdot \frac{\partial F_i(x,t,z_x(x,T))}{\partial z_{x_j}} \cdot \cos(\nu, x_j) = 0, \quad x \in \partial\Omega .$$

$$H_u(x,t,z,z_x,z_t,\psi,u) = \Phi(x,t,z,z_x,z_t,u) + \psi f_1(x,t,u).$$

In section **1.2** of this chapter necessary conditions for optimality imposed on the problem in section 1.1are derived.

Theorem 2. (Integral condition for optimality) Let the conditions of theorem 1 be fulfilled and U be a convex set. If $u^0(x,t)$ is an optimal control, then for each admissible control $u(x,t)$ we have the inequality

$$\int_{\Pi_\alpha} (H_u(x,t), u(x,t) - u^0(x,t))dxdt \geq 0,$$

where

$$\Pi_\alpha = \{x_i^0 - \sqrt[n+1]{\alpha} \leq x_i \leq x_i^0 + \sqrt[n+1]{\alpha},\ t^0 - \sqrt[n+1]{\alpha} \leq t \leq$$
$$\leq t^0 + \sqrt[n+1]{\alpha},\ i = 1,2,...,n\}$$

is an $(n+1)$ dimensional parallelepiped contained in Q, $H_u(x,t)$ is a derivative of $H_u(x,t,z,z_x,z_t,\psi,u)$ for $u = u^0(x,t)$, and $z^0(x,t), \psi^0(x,t)$ is the solution of the problem (2), (3) and of the conjugate problem corresponding to $u = u^0(x,t)$.

Then, using the property of the convex functional we prove the following theorem.

Theorem 3. (Necessary and sufficient conditions for optimality). Let U be a convex set, $I(u)$ be a convex functional and the conditions of theorem 1 be fulfilled. For the optimality of the control $u^0(x,t)$ it is necessary and sufficient that the following condition be fulfilled

$$\min_{u \in U} \int_Q (H_u(x,t), u - u^0(x,t))dxdt = 0.$$

Chapter II consists of two sections.

In section **2.1** we consider the following problem: find such control functions $u = u(x,y,t),\ v = v(x,y)$ that together with appropriate solution of the initial boundary value problem

$$\beta z_{tt} + z_t - \frac{\partial}{\partial t}\Delta z - \Delta z = u(x,y,t),\ (x,y) \in Q\,, \tag{4}$$

$$z(0,y,t) = z(x,0,t) = z(a,y,t) = z(x,b,t) = 0\,, \tag{5}$$

$$z(x,y,0) = v(x,y),\ z_t(x,y,0) = 0\,, \tag{6}$$

affords a minimum to the functional

$$J(u,v) = \int_\Omega \left| z(x,y,T) - z_0(x,y) \right|^2 dxdy +$$

$$+ \alpha_1 \int_Q u^2(x,y,t)dxdydt + \alpha_2 \int_\Omega v^2(x,y)dxdy\,, \tag{7}$$

where $Q = (0,T) \times \Omega = \{0 < t < T,\ 0 < x < a, 0 < y < b\}$, β, T, a, b are the given positive numbers, $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplace operator.

According to the generally accepted, $u(x,y,t)$ is called a distributed control, and following Lions, we call $v(x,y)$ a starting control.

As a distributed control we take the functions $u(x,y,t) \in L_2(Q)$ for which $\|u\|_{L_2} \le R^2$, as a starting control $v(x,y)$ the functions from $\overset{0}{W}{}_2^1(\Omega) \cap W_2^2(\Omega)$, moreover $\left|\frac{\partial v}{\partial x}\right| \frac{\partial v}{\partial x}, \left|\frac{\partial v}{\partial y}\right| \frac{\partial v}{\partial y} \in W_2^1(\Omega)$, for which almost everywhere in Ω the conditions $l_1 \le v(x,y) \le l_2$ are fulfilled. Then for the admissible pair $p = (u(x,y,t), v(x,y))$ the Hilbert space $H = L_2(Q) \times W_2^1(\Omega)$ with a scalar product

$$\langle p_1, p_2 \rangle = \int_Q u_1(x,y,t) u_2(x,y,t) dxdydt +$$

$$+ \int_\Omega \left[v_1(x,y) v_2(x,y) + \frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial y} \right] dxdy$$

is introduced and definition of the generalized solution (4)-(6) is given:

Under the generalized solution we understand the function

$$z \in L_2\left(0,T;\overset{0}{W}{}_2^1(\Omega)\right), z_t \in L_\infty(0,T;L_2(\Omega)) \cap L_2\left(0,T;\overset{0}{W}{}_2^1(\Omega)\right),$$

$$B^{\frac{1}{2}}\Delta z \in L_2(0,T;L_2(\Omega)),$$

satisfying the integral identity

$$\int_Q (\beta z_t(x,y,t) + z(x,y,t))g(x,y,t)dxdydt +$$

$$+\int_Q (z_x(x,y,t)g_x(x,y,t) + z_y(x,y,t)g_y(x,y,t))dxdydt +$$

$$+\int_Q g_x(x,y,t)\int_0^t z_x(x,y,s)dsdxdydt +$$

$$+\int_Q g_y(x,y,t)\int_0^t z_y(x,y,s)dsdxdydt = \int_Q v(x,y)g(x,y,t)dxdydt +$$

$$+\int_Q v_x(x,y)g_x(x,y,t)dxdydt + \int_Q v_y(x,y)g_y(x,y,t)dxdydt +$$

$$+\int_Q g(x,y,t)\int_0^t u(x,y,s)dsdxdydt,$$

where $B(x)$ is a smooth function positive inside Ω, while $g(x,y,t)$ is an arbitrary function from $L_2\left(0,T;\overset{0}{W}{}_2^1(\Omega)\right)$.

After transforming the increment it is shown that the functional $J(u,v)$ is differentiable in the introduced space H and from which the validity of the theorem follows.

Theorem 4. Functional (7) is differentiable in H and its gradient

$$J'(u,v) = (-\psi(x,y,t) + 2\alpha_1 u(x,y,t); 2[z(x,y,T) - z_0(x,y)] +$$

$$+\int_0^T \Delta\psi(x,y,s)ds + 2\alpha_2 v(x,y)),$$

where $\psi(x,y,t)$ is the solution of the conjugate problem:

$$-\beta\psi_t+\psi-\Delta\psi+\int_t^T\Delta\psi(x,y,s)ds+2[z(x,y,T)-z_0(x,y)]=0,$$

$$\psi(x,y,T)=0,\qquad \psi(x,y,t)\,|_{\partial\Omega}=0,\ 0\le t\le T.$$

Since equation (4) and conditions (5), (6) are linear, quadratic functional (7) and the set of admissible controls are convex, from the known property of (8) we get the validity of the following theorem.

Theorem 5. For optimality of the control $p^0(x,y,t)=(u^0(x,y,t),v^0(x,y))$ it is necessary and sufficient the inequality

$$\int_Q(-\psi(x,y,t)+2\alpha_1u^0(x,y,t))(u(x,y,t)-u^0(x,y,t))dxdydt+$$

$$+\int_\Omega\left[2(z(x,y,T)-z_0(x,y))+\int_0^T\Delta\psi(x,y,s)ds+2\alpha_2v^0(x,y)\right]\times$$

$$\times(v(x,y)-v^0(x,y))dxdy\ge 0$$

be fulfilled for any $p(x,y,t)=(u(x,y,t),v(x,y))$.

In section **2.2.** of this chapter we study the following problem: find such an admissible control $(v^0(x,y),v^1(x,y),u(x,y,t))$, that the solution of the equation (4) corresponding to it satisfying boundary conditions (5) and initial conditions

$$z(x,y,0)=v^0(x,y),\ z_t(x,y,0)=v^1(x,y) \tag{9}$$

satisfy the condition

$$z(x,y,T)=\varphi(x,y), \tag{10}$$

and the functional

$$J(u,v^0,v^1)=\int_Q u^2(x,y,t)dxdydt+$$

$$+\alpha_0\int_\Omega\left(v^0(x,y)\right)^2dxdy+\alpha_1\int_\Omega\left(v^1(x,y)\right)^2dxdy \tag{11}$$

take the least value, where $\varphi(x,y)$ is a given function from $L_2(\Omega)$ and α_0, α_1 are positive constants.

Herewith, as a distributed control we take the functions $u(x,y,t)\in L_2(Q)$, for which $\|u\|_{L_2(Q)}\le R$, while as starting controls we take the functions $v^0(x,y) \in W_2^2(\Omega)$, for which $l_1^0 \le v^0 \le l_2^0$ and the functions $v^1(x,y) \in L_2(\Omega)$, for which $l_1^1 \le v^1 \le l_2^1$.

At first we introduce definition of the generalized solution of the problem (4), (5), (9), under which we understand the function

$z(x,y,t) \in L_2(0,T,\overset{0}{W}{}_2^1(\Omega))$; $z_t \in L_\infty(0,T,L_2(\Omega)) \cap L_2(0,T,\overset{0}{W}{}_2^1(\Omega))$, $B^{\frac{1}{2}}\Delta z \in L_2(0,T,L_2(\Omega))$

satisfying the integral identity

$$\int_Q [\beta z_t(x,y,t) + z(x,y,t)] g(x,y,t)\,dxdydt +$$

$$+\int_Q [z_x(x,y,t) g_x(x,y,t) + z_y(x,y,t) g_y(x,y,t)]\,dxdydt +$$

$$+\int_Q g_x(x,y,t)\int_0^t z_x(x,y,s)\,dsdxdydt + \int_Q g_y(x,y,t)\int_0^t z_y(x,y,s)\,dsdxdydt =$$

$$= \int_Q [\beta v^1(x,y) + v^0(x,y)] g(x,y,t)\,dxdydt +$$

$$+\int_Q [v_x^0(x,y) g_x(x,y,t) + v_y^0(x,y) g_y(x,y,t)]\,dxdydt +$$

$$+\int_Q g(x,y,t)\int_0^t u(x,y,s)\,dsdxdydt,$$

where $B(x,y)$ is a smooth function positive inside Ω and $g(x,y,t)$ is an arbitrary function from $L_2(0,T,\overset{0}{W}{}_2^1(\Omega))$, and then applying the method of separation of variables (i.e. the Fourier method) we obtain the representation of the solution of the problem in the from of a double series

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{k_2(m,n) - k_1(m,n)} \left\{ \frac{1}{\beta}\int_0^t \left[e^{k_2(m,n)(t-s)} - e^{k_1(m,n)(t-s)}\right] u_{mn}(s)\,ds + \right.$$

$$+\frac{4}{ab}\int_0^a\int_0^b \left[\left(k_2(m,n)e^{k_1(m,n)t} - k_1(m,n)e^{k_2(m,n)t}\right)v^0(\xi,\eta) + \left(e^{k_2(m,n)t} - e^{k_1(m,n)t}\right)v^1(\xi,\eta)\right] \times$$

$$\times \sin\frac{m\pi\xi}{a}\cdot\sin\frac{n\pi\eta}{b}d\xi d\eta\Big\}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}, \tag{12}$$

where

$$\lambda_{mn}=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2,\ X_{mn}(x,y)=\frac{2}{\sqrt{ab}}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b},\quad m,n=1,2,...$$

are eigen values and eigen functions of the problem

$$\Delta X+\lambda X=0,\quad X|_{\Gamma}=0,$$

where Γ is the boundary of the rectangle Ω; $k_1(m,n)$, $k_2(m,n)$ are the roots of the characteristical equation for the differential equation

$$\beta\ddot{T}+(\lambda_{mn}+1)\dot{T}+\lambda_{mn}T=0\quad m,n=1,2,...,$$

$$u_{mn}(t)=\frac{4}{ab}\int_0^a\int_0^b u(x,y,t)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dxdy$$

Assume

$$v^0(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}v_{mn}^0\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b},$$

$$v^1(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}v_{mn}^1\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}.$$

It is proved that the series in (12) and the series obtained from (12) by differentiation with respect to x,y,t converge strongly in L_2 and therefore the expression (12) is the solution of (4), (5), (9). Then the functional minimum problem (11) is reduced to the problem more exactly to the sequence of problems for the conditional extremum of the functional

$$I_{mn}=\int_0^T u_{mn}^2(t)dt+\alpha_0\left(v_{mn}^0\right)^2+\alpha_1\left(v_{mn}^1\right)^2 \tag{13}$$

provided

$$\int_0^T\left[e^{k_2(m,n)(T-s)}-e^{k_1(m,n)(T-s)}\right]u_{mn}(s)ds+\beta\left(k_2(m,n)e^{k_1(m,n)T}-k_1(m,n)e^{k_2(m,n)T}\right)v_{mn}^0+$$

$$+\beta\left(e^{k_2(m,n)T}-e^{k_1(m,n)T}\right)v_{mn}^1=\beta k_2\left((m,n)-k_1(m,n)\right)\varphi_{mn}\quad m,n=1,2,..., \tag{14}$$

where

$$\varphi_{mn} = \frac{4}{ab}\int_0^a\int_0^b \varphi(x,y)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\,dxdy\,.$$

For this problem the Lagrange function is complied, its stationary point $\left(\tilde{u}_{mn}(t), \tilde{v}_{mn}^0, \tilde{v}_{mn}^1\right)$ is determined and it is proved that this point is a point of the minimum of the functional (13) subject to the condition (14).

Then we prove

Theorem 6. If $\varphi(x,y) \in L_2(\Omega)$, then the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{u}_{mn}^2(t)\,,\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\tilde{v}_{mn}^1\right)^2\ ,\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\tilde{v}_{mn}^1\right)^2$$

converge in $L_2(\Omega)$, $W_2^2(Q)$ and $L_2(\Omega)$, respectively.

Chapter III of the dissertation consists of two sections. In section **3.1** we consider the following minimal energy stabilization problem: to find such an admissible control $u = u(x,y,t)$ for which the solution of the equation, (4) satisfying the condition (5) and the condition

$$z(x,y,0) = z_0(x,y), \quad z_t(x,y,0) = z_1(x,y) \tag{15}$$

satisfies the condition

$$z(x,y,T) = 0 \tag{16}$$

and the functional

$$J = \| u \|^2 = \int_Q u^2(x,y,t)\,dxdydt$$

takes the least value.

For solving the stated problem, at first by means of the Fourier transform we obtain the representation of the solution of the initial boundary value problem in the form

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\beta(k_2(m,n)-k_1(m,n))}\cdot$$

$$\cdot\left\{\int_0^t\left[e^{k_2(m,n)(t-s)} - e^{k_1(m,n)(t-s)}\right]u_{mn}(s)\,ds + \right.$$

$$+\frac{4}{ab}\cdot\int_0^a\int_0^b\Big[\big(k_2(m,n)z_0(\xi,\eta)-z_1(\xi,\eta)\big)e^{k_1(m,n)t}+$$

$$+\big(z_1(\xi,\eta)-k_1(m,n)z_0(\xi,\eta)\big)e^{k_2(m,n)t}\Big]\times$$

$$+\frac{4}{ab}\cdot\int_0^a\int_0^b\Big[\big(k_2(m,n)z_0(\xi,\eta)-z_1(\xi,\eta)\big)e^{k_1(m,n)t}+\big(z_1(\xi,\eta)-k_1(m,n)z_0(\xi,\eta)\big)e^{k_2(m,n)t}\Big]\times$$

$$\times\sin\frac{m\pi\xi}{a}\cdot\sin\frac{n\pi\eta}{b}d\xi d\eta\Big\}\times\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

and show that the obtained series and the series obtained by replacement of its members by the derivatives with respect to t,x,y converge uniformly in $L_2(Q)$. Consequently, this series is generalized solution of the problem (4), (5), (15). Then by means of this representation and the condition $z(x,y,T)=0$ we obtain the equations

$$\int_0^T\frac{e^{k_2(m,n)(T-s)}-e^{k_1(m,n)(T-s)}}{\beta(k_2(m,n)-k_1(m,n))}u_{mn}(s)ds=f_{mn},\ \ m,n=1,2,... \qquad (17)$$

for determining $u_{mn}(t)$, where

$$u_{mn}(t)=\frac{4}{ab}\int_0^a\int_0^b u(x,y,t)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dxdy, m,n=1,2,...$$

$$f_{mn}(t)=\int_0^a\int_0^b[-W(x,y,T)]\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dxdy,$$

while $W(x,y,t)$ is the solution of the homogeneous equation corresponding to (4) under the conditions

$$W(0,y,t)=W(x,0,t)=W(a,y,t)=W(x,b,t)=0,$$

$$W(x,y,0)=z_0(x,y),W_t(x,y,0)=z_1(x,y).$$

Thus, the problem of minimum of the functional $J(u)$ is reduced to the definition of the sequence $\{u_{mn}(t),\ m,n=1,2,...\}$ satisfying the system of equations (17) for which the functional

$$J=\int_0^T\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}^2(t)dt=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}J_{mn} \qquad (18)$$

takes the least value.

It is clear that the obtained problem is a moments problem and it is shown that it has a unique solution

$$v_{mn}^{0}(t)=\frac{f_{mn}}{\alpha_{mn}}F_{mn}(T-t) \ ,$$

where

$$F_{mn}(T-t)=\frac{e^{k_2(m,n)(T-t)}-e^{k_1(m,n)(T-t)}}{\beta(k_2(m,n)-k_1(m,n))},\ \alpha_{mn}=\int_0^T F_{mn}^2(T-t)dt$$

and for this $v_{mn}^{0}(t)$ the least value J_{mn} will be

$$J_{mn}=\frac{f_{mn}^2}{\int_0^T F_{mn}^2(T-t)dt},$$

while

$$J=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}J_{mn}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{f_{mn}^2}{\int_0^T F_{mn}^2(T-t)dt}. \tag{19}$$

In the end of the section we prove the convergence of the double series (19).

Thus, we have the following theorem

Theorem 7. If $z_0(x,y)\in \overset{0}{W}{}_4^1(\Omega)\cap W_2^2(\Omega)$, $\left|\frac{\partial z_0}{\partial x}\right|\frac{\partial z_0}{\partial x}$ and $\left|\frac{\partial z_0}{\partial y}\right|\frac{\partial z_0}{\partial y}$ from $W_2^1(\Omega)$, $z_1(x,y)\in L_2(Q)$, then the double series (19) converges.

In section **3.2** of this chapter we consider the following problem: to find such an admissible control $u=u(x,y,t)$ that corresponding solution of equation (4) satisfying the condition (5) and the condition

$$z(x,y,0)=0,\quad z_t(x,y,0)=z_1(x,y),\ (x,y)\in\Omega, \tag{20}$$

satisfies the condition

$$z_t(x,y,T)=0,(x,y)\in\Omega, \tag{21}$$

while th functional

$$J=\int_Q u^2(x,y,t)dxdydt$$

takes the least value.

First of all, the solution of the initial boundary value problem (4), (5), (20) is sought in the form

$$z(x,y,t)=V(x,y,t)+W(x,y,t),$$

where $V(x,y,t)$ is the solution of the inhomogeneous equation

$$\beta V_{tt}+V_t-\frac{\partial}{\partial t}\Delta V-\Delta V=u(x,y,t) \tag{22}$$

satisfying the conditions

$$V(0,y,t)=V(x,0,t)=V(a,y,t)=V(x,b,t)=0, \tag{23}$$

$$V(x,y,0)=0,\ \ V_t(x,y,0)=0, \tag{24}$$

while $W(x,y,t)$ is the solution of the homogeneous equation

$$\beta W_{tt}+W_t-\frac{\partial}{\partial t}\Delta W-\Delta W=0, \tag{25}$$

satisfying the boundary conditions

$$W(0,y,t)=W(x,0,t)=W(a,y,t)=W(x,b,t)=0 \tag{26}$$

and the initial condition

$$W(x,y,0)=0,\ \ W_t(x,y,0)=z_1(x,y). \tag{27}$$

At first, applying the method of separation of variables (the Fourier method) we obtain the representation of the problem solution and then solve the problem (22), (23), (24) and by their means obtain the representation

$$z(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k_2(m,n)-k_1(m,n)}\frac{1}{\beta}\int_0^t\left[e^{k_2(m,n)(t-s)}-e^{k_1(m,n)(t-s)}\right]u_{mn}(s)ds+$$

$$+\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a\int_0^b\left\{\frac{-z_1(\xi,\eta)e^{k_1(m,n)t}}{k_2(m,n)-k_1(m,n)}+\frac{z_1(\xi,\eta)e^{k_2(m,n)t}}{k_2(m,n)-k_1(m,n)}\right\}\times$$

$$\times\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}d\xi d\eta\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{28}$$

of the solution of the problem (4), (5), (20) and prove the convergence of the series in the right hand side of this representation

Using the representation (28) by means of the condition (21) the minimum problem is reduced to the sequence of problems on conditional extremum of functionals

$$I_{mn} = \int_0^T u_{mn}^2(t)dt, \quad m,n = 1,2,...$$

under the conditions

$$\int_0^T \left[k_2(m,n)e^{k_2(m,n)(T-s)} - k_1(m,n)e^{k_1(m,n)(T-s)}\right] u_{mn}(s)ds =$$

$$= -\beta\left[k_2(m,n)e^{k_2(m,n)T} - k_1(m,n)e^{k_1(m,n)T}\right] z_{1mn},\ m,n = 1,2,....$$

Then we find stationary points of these problems in the form

$$\tilde{u}_{mn}(t) = -\frac{\beta B_{mn}(0) z_{1mn} B_{mn}(t)}{\int_0^T B_{mn}^2(t)dt},$$

$$B_{mn}(t) = k_2(m,n)e^{k_2(m,n)(T-t)} - k_1(m,n)e^{k_1(m,n)(T-t)}$$

and prove the convergence of the double series

$$J = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^T u_{mn}^2(t)dt,$$

more exactly, prove the following theorem

Theorem 8. The function $u(x,y,t)$ determined by the series

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{u}_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

affords a minimum to the functional and the corresponding solution of the problem (4), (5), (20) satisfies the condition (21).

CONCLUSION

The basic results of the dissertation work are in the following works:

1. Ягубова, М.М. Задачи оптимального управления в процессах, описываемых уравнением с частными производными третьего порядка // -Баку: Вестник Бакинского Университета, сер. физ.-мат. наук, -2015, №2, -с.77-89
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