REPUBLIC OF AZERBAIJAN

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FUZZY LOGIC'S Z-EXTENSION-BASED DECISION TOOLS AND THEIR APPLICATIONS

Specialty: 3338.01 – Systems analysis

Field of science: Technical

Applicant: **Rafig Aliyev**

A B S T R A C T

of the dissertation for the degree of Doctor of Philosophy

Baku - 2021

Dissertation was accomplished at the research laboratory "Intelligent Control and Decision Making Systems in Industry and Economics" of Azerbaijan State Oil and Industry University.

Scientific supervisor	Doctor of technical sciences, prof. **Latafat A. Gardashova**
Official opponents	1. Doctor of technical sciences, prof. **Salahaddin I. Yusifov** 2. Doctor of technical sciences, prof. **Mahammad N. Nuriyev** 3. Doctor of technical sciences, prof. **Valeh A. Mustafayev**

Dissertation council ED 2.02 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Azerbaijan State Oil and Industry University

Chairman of the Dissertation Council:	Doctor of technical sciences, prof. **Mustafa B. Babanli**
Scientific secretariat of the Dissertation Council:	Doctor of philosophy on technical sciences, assoc. prof. **Tahir G. Jabbarov**
Chairman of the Scientific Seminar	Doctor of technical sciences, prof. **Tarlan S. Abdullayev**

GENERAL DESCRIPTION OF THE DISSERTATION

The actuality of the topic. Analyzing the existing decision theories we arrived at conclusion that these theories are developed for a decision environment characterized by well-described information on alternatives, states of nature, probabilities and outcomes.

It is very well known that validity and effectiveness of decision analysis mainly is related with consistent formulation of decision maker's (DM) preferences. In real-world problems, DM's knowledge is inherently associated with imprecision and partial reliability. This involves combination of fuzzy and probabilistic information. The concept of a Z-number is a formal construct to describe such kind of information. In this study, we formulate the concepts of Z-number-valued eigenvalue and eigenvector for matrices components of which are Z-numbers. Consequently, there is need to investigate consistency of a DM's preference knowledge which is related to eigenvalues and eigenvectors of decision matrices.

The notion of consistency is used to estimate the quality of preference knowledge and its stability for reliable evaluation of decision alternatives. In the famous AHP method there is a set of strict consistency conditions used to keep the rationality of preference intensities between compared elements. These requirements are not achievable in the real situations when DM has limited rationality and partially reliable preferences. A new approach to deriving consistency-driven preference degrees for such kind of situations is a research problem.

Decision making is based on preferences over alternatives and choice criteria. Relation equations are used to formalize imprecise information about dependence of variables of interest sourced from human knowledge. However, real-world information is also characterized by partial reliability of the sources. One of the important formal constructs developed to deal with a fusion of fuzziness and partial reliability of information is the Z-number concept.

One of the main branches of decision theory is multi-attribute decision making (MADM). The process of MADM is to find the best option among all of the existing alternatives. The use of one or another multi-attribute decision theory depends mainly on decision making situations. Most of existing techniques is not much effective for solving MADM problems under decision information characterized by fuzziness and partial reliability. Unfortunately, existing works on MADM under bimodal (fuzzy + probability) information is very scarce.

In this dissertation we intend to investigate and solve the considered problems.

Goal of the dissertation. The goal of this thesis is to create methods and methodologies for construction of consistent and adequate preferences of a DM characterized by imprecise and partially reliable knowledge. Also, the goal is to propose a new approach to decision making in uncertain environment based on Z-relation (bimodal relation). Two new approaches to MADM problem based on ideal solution and similarity measure are included to the goal.

Main highlights, brought forward for dissertation defense.

In dissertation the problems shown below are considered:

- review of existing decision making theories and their critical analysis;

- construction of consistency-driven decision preferences, characterized by imprecision and partial reliability and investigation of eigensolutions of preference matrices;

-introduction of concept of Z-relation equation and development of Z-relation-based decision making method;

- development of Z-valued MADM methods on basis of distance and similarity concepts.

Research methods. Research theories, methods and methodologies used in this dissertation are fuzzy logic, Z-extension of fuzzy logic, computation with fuzzy and Z-numbers, eigenvalue and eigenvector analysis, fuzzy relation equations, distance and similarity measures.

Scientific value of the thesis. Obtained theoretical results on analysis of consistency-driven decision preferences on basis of decision matrices with Z-valued elements (imprecise and partially reliable elements) and computation of eigensolutions are new. Scientific contribution to decision making theories is related to introduction of Z-relation definition, operations with Z-relations and Z-relation equations, and development of Z-relation-based decision-making method. Also, the suggested distance-based and similarity-based MADM methods under Z-decision environment are new and can be friendly used by practitioners.

Practical value of the thesis. Mentioned above theoretical findings are universal and can be used in economics, forecasting, engineering, business, risk analysis etc.

Realization of results of dissertation. Research results of the dissertation were applied in decision analysis in different problems, including country selection for business location, hotel business decision making, web services comparison and project selection problems.

Approbation of dissertation. Main results of dissertation were presented in the international conferences:

- ***ICSCCW-2009*** 5th International Conference on Soft Computing and Computing with Words and Perceptions, Famagusta, North Cyprus.
- ***ICAFS-2010*** 9th International Conference on Application of Fuzzy Systems and Soft Computing, Prague, Czech Republic.
- ***ICSCCW-2011*** 6th International Conference on Soft Computing and Computing with Words and Perceptions, Antalya, Turkey.
- ***ICAFS-2012*** 10th International Conference on Application of Fuzzy Systems and Soft Computing, Lisbon, Portugal.
- ***WCIS-2012*** 7th World Conference on Intelligence Systems for Industrial Automation, Tashkent, Uzbekistan.
- ***ICSCCW-2015*** 8th International Conference on Soft Computing and Computing with Words and Perceptions, Antalya, Turkey.
- ***ICAFS-2016*** 12th International Conference on Application of Fuzzy Systems and Soft Computing, ICAFS 2016, 29-30 August 2016, Vienna, Austria

- ***ICSCCW-2017*** 9th International Conference on Theory and Application of Soft Computing, Computing with Words and Perceptions – ICSCCW-2017, 22-23 August 2017, Budapest, Hungary
- ***ICAFS-2018*** 13th International Conference on Theory and Application of Fuzzy Systems and Soft Computing - ICAFS-2018, 26-27 August 2018, Warsaw, Poland
- ***ICAFS-2020*** 14th International Conference on Theory and Application of Fuzzy Systems and Soft Computing – ICAFS-2020, 27-28 August 2020, Budva, Montenegro
- ***WCIS-2020*** 11th World Conference on Intelligent systems for industrial automation – WCİS-2020, 26-28 November, Tashkent, Uzbekistan.

Organization where dissertation was realized: Azerbaijan State Oil and Industry University, Research laboratory "Intelligent Control and Decision Making Systems in Industry and Economics".

Structure of dissertation. Manuscript of dissertation includes introduction, 6 chapters, conclusion, and references.

Publications. Although author has 16 published works, the obtained results in dissertation were published in 13 works including 6 works in SCIE, 8 works in Web of Science Core Collection, 9 works in SCOPUS database, and 3 works in Conference Proceedings.

In the first chapter the state of the art of decision theories, their critical analysis, and some preliminary information on Z-extension of fuzzy logic are given.

Comprehensive critical analysis of existing decision theories and tools has shown that effective research results on decision making methods under imprecise and partially reliable decision information are very scarce. In particular, in scientific literature there are no works on formulation of more adequate decision preferences characterized by bimodal information, i.e. information containing synergy of fuzzy and probabilistic uncertainties. The concept of Z-number - the Z-extension of fuzzy logic is a formal construct to describe such kind of information.

Continuous Z-number[1,2]

A continuous Z-number is an ordered pair $Z=(A,B)$ where A is a continuous fuzzy number playing a role of a fuzzy constraint on values that a random variable X may take:

$$X \ is \ A, \tag{1}$$

B is a continuous fuzzy number with a membership function $\mu_B:[0,1]\to[0,1]$, playing a role of a fuzzy constraint on the probability measure of A:

$$P(A)=\int_R \mu_A(x)p(x)dx \ is \ B. \tag{2}$$

Formula (2) implies that actual distribution *p(x)* is not known. A concept of a continuous Z^+-number is closely related to the concept of a continuous Z-number[1]. Basically, a Z^+-number is a combination of a fuzzy number, A, and a random number, R, written as an ordered pair $Z^+\equiv(A,R)$. In this pair, A plays the same role as it does in a Z-number, and R is some probability distribution of a random number. A Z^+-number is associated with a bimodal distribution as a combination the possibility and probability distributions.

A discrete Z-number[3]

A discrete Z-number is an ordered pair $Z=(A,B)$ where A is a discrete fuzzy number which describes a fuzzy constraint on values that a random variable X may take – "$X \ is \ A$", and B is a discrete fuzzy number with a membership function $\mu_B:\{b_1,...,b_n\}\to[0,1]$, $\{b_1,...,b_n\}\subset[0,1]$, which describes a fuzzy constraint on the probability measure of A:

$$P(A) \ is \ B \tag{3}$$

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[1] Zadeh L.A. A note on Z-numbers. Information Sciences, 181(14), pp. 2923–2932 (2011).

[2] Aliev R.A. Uncertain computation-based decision theory, World Scientific, (2017).

[3] Aliev R.A., Huseynov O.H., Aliyev R.R., Alizadeh A.V. The Arithmetic of Z-numbers. Theory and Applications, World Scientific (2015).

A Distance between Z-numbers[4]

As a Z-number $Z=(A,B)$ is characterized by fuzzy number *A*, fuzzy number *B* and underlying set of probability distributions *G*, the distance between Z-numbers $D(Z_1,Z_2)$ is defined as follows.

Distance between A_1 and A_2 is computed as

$$D(A_1,A_2)=\sup_{\alpha\in(0,1]} D(A_1^\alpha,A_2^\alpha) \tag{4}$$

where

$$D(A_1,A_2)=\left|\frac{A_{11}^\alpha+A_{12}^\alpha}{2}-\frac{A_{21}^\alpha+A_{22}^\alpha}{2}\right| \tag{5}$$

Distance between B_1 and B_2 is computed analogously.

It is needed to find distance between the sets G_1 and G_2 of probability distributions p_1 and p_2 underlying Z_1 and Z_2. The distance between p_1 and p_2 can be expressed as

$$D(G_1,G_2)=\inf_{p_1\in G_1,\,p_2\in G_2}\left\{(1-\int_R (p_1p_2)^{\frac{1}{2}}dx)^{\frac{1}{2}}\right\} \tag{6}$$

Given $D(A_1,A_2)$, $D(B_1,B_2)$ and $D(G_1,G_2)$, the distance for Z-numbers is defined as

$$D(Z_1,Z_2)=\beta D(A_1,A_2)+(1-\beta)D_{total}(B_1,B_2), \tag{7}$$

where $D_B^{total}(B_1,B_2)$ is a distance for reliability restriction computed as

$$D_B^{total}(B_1,B_2)=wD(B_1,B_2)+(1-w)D(G_1,G_2) \tag{8}$$

$\beta,w\in[0,1]$ are DM’s assigned importance degrees.

Similarity of Z-numbers[5]

A similarity $S(Z_1,Z_2)$ of Z-numbers $Z_1=(A_1,B_1)$ and $Z_2=(A_2,B_2)$ is defined as follows:

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[4] Aliev R.A., Pedrycz W., Huseynov O.H., Aliyev R.R. Eigensolutions of partially reliable decision preferences described by matrices of Z-numbers. International of Journal Information Technology Decision Making 19(6), pp 1429-1450 (2020).

[5] Aliyev R.R. Similarity based multi-attribute decision making under Z-information. b-Quadrat Verlag, p.33-39 (2015).

$$S(Z_1, Z_2) = \frac{1}{1 + D(Z_1, Z_2)} \tag{9}$$

where D is distance between Z-numbers.

Ranking of Z-numbers

For Z-numbers Z, Z' it holds:

$$Z \leq Z' \text{ iff } D(Z, (1,1)) \geq D(Z', (1,1)) \tag{10}$$

where D is distance defined above, (1,1) is a fuzzy singletons-based Z-number.

One can easily prove that $\leq$ is a partial order as it poses the following properties:

$$\begin{gathered} Z \leq Z \text{ (reflexivity)} \\ \text{If } Z \leq Z' \text{ and } Z' \leq Z \text{ then } Z = Z' \text{ (antisimmetry)} \\ \text{If } Z \leq Z' \text{ and } Z' \leq Z'' \text{ then } Z \leq Z'' \text{ (transitivity).} \end{gathered} \tag{11}$$

In the second chapter some operations on Z-numbers are considered. Basically, a Z^+-number, Z^+, is a combination of a fuzzy number, *A*, and a random number, *R*, written as an ordered pair $Z^+ \equiv (A; R)$. In this pair, *A* plays the same role as it does in a *Z*-number, and *R* is the probability distribution of a random number.

R may be viewed as the probability distribution of *X* in the *Z*-valuation (X, A, B). A Z^+-number may be expressed as (A, p) or, equivalently, (μ_A, p_X), where μ_A is the membership function of *A*. Then a Z^+-valuation is expressed as (X, A, p_X) (or as (X, μ_A, p_X), where p is the probability density over *X*. A Z^+-number is associated with a bimodal distribution as a combination the possibility and probability distributions over *X*.

When distributions p in Z-number are taken as some parametric distributions, e.g. normal distribution, the operations over Z-numbers are sufficiently simplified[1]. The density function of a normal distribution is

$$p_U(u) = normpdf(u, m, \sigma)) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) \tag{12}$$

In this situation, for any m, σ we have

$$\begin{aligned} &\text{Prob}_{m,\sigma}(U\ is\ A) = \int_{-\infty}^{+\infty} \mu_A(u) p_{m,\sigma}(u) du = \int_{-\infty}^{+\infty} \mu_A(u) \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{(u-m)^2}{2\sigma^2}\right) du = \\ &= quad(trapmf(u, [a_1, a_{2,}, a_3, a_4]) * normpdf(u, m, \sigma), -\text{inf}, +\text{inf}) \end{aligned} \tag{13}$$

Then the space $\mathbf{P}$ of probability distributions will be the class of all normal distributions each uniquely defined by its parameters m, σ.

Let $U = (A_U, B_U)$ and $V = (A_V, B_V)$ be two independent Z-numbers. Consider determination of $W = U + V$. First, we need compute $A_U + A_V$ using Zadeh's extension principle:

$$\mu_{(A_U + A_V)}(w) = \sup_u (\mu_{A_U}(u) \wedge \mu_{A_V}(w-u)), \quad \wedge = \min \tag{14}$$

As the sum of random variables involves the convolution of the respective density functions we can construct $\tilde{P}_W$, the fuzzy subset of $\mathbf{P}$, associated with the random variable W[6]. Recall that the convolution of density functions p_1 and p_2 is defined as the density function

$$p = p_1 \oplus p_2 \tag{15}$$

such that

$$p(w) = \int_{-\infty}^{+\infty} p_1(u) p_2(w-u) du = \int_{-\infty}^{+\infty} p_1(w-u) p_2(u) du \tag{16}$$

One can then find the fuzzy subset $\tilde{P}_W$. For any $p_W \in \mathbf{P}$, one obtains

$$\mu_{P_W}(p_W) = \max_{p_U, p_V} [\mu_{P_U}(p_U) \wedge \mu_{P_V}(p_V)], \tag{17}$$

subject to

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[1] Zadeh L.A. A note on Z-numbers. Information Sciences, 181(14), pp. 2923–2932 (2011).

[6] Papoulis A. Probability, Random Variables, and Stochastic Processes, McGraw-Hill, New York (1965).

$p_W = p_U \oplus p_V$, that is, $p_W(w) = \int_{-\infty}^{+\infty} p_U(u) p_V(w-u)du = \int_{-\infty}^{+\infty} p_U(w-u) p_V(u)du$.

Given $\mu_{P_U}(p_U) = \mu_{P_U}(m_U, \sigma_U)$ and $\mu_{P_V}(p_V) = \mu_{P_V}(m_V, \sigma_V)$ as

$$\mu_{P_U}(m_U, \sigma_U) = \mu_{B_U}(\int_{-\infty}^{+\infty} \mu_{A_U}(u) \frac{1}{\sigma_U \sqrt{2\pi}} \exp\left(\frac{(u-m_U)^2}{2\sigma_U{}^2}\right) du \tag{18}$$

$$\mu_{P_V}(m_V, \sigma_V) = \mu_{B_V}(\int_{-\infty}^{+\infty} \mu_{A_V}(u) \frac{1}{\sigma_V \sqrt{2\pi}} \exp\left(\frac{(u-m_V)^2}{2\sigma_V{}^2}\right) du \tag{19}$$

one can define $\tilde{P}_W$ as follows

$$p_W = p_{m_U, \sigma_U} \oplus p_{m_V, \sigma_V} , \tag{20}$$

$$\begin{aligned} p_W(w) &= p_{m_W, \sigma_W} = normpdf[w, m_W, \sigma_W] = \\ &= quad(normpdf(u, m_U, \sigma_U) * normpdf(w-u, m_V, \sigma_V), -\text{inf}, +\text{inf}) = \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sigma_U \sqrt{2\pi}} \exp\left(\frac{(u-m_U)^2}{2\sigma_U{}^2}\right) \frac{1}{\sigma_V \sqrt{2\pi}} \exp\left(\frac{(w-u-m_V)^2}{2\sigma_V{}^2}\right) du \end{aligned} \tag{21}$$

where

$$m_W = m_U + m_V \text{ and } \sigma_W = \sqrt{\sigma_U^2 + \sigma_V^2} , \tag{22}$$

$$\mu_{P_W}(p_W) = \sup(\mu_{P_U}(p_U) \wedge \mu_{P_V}(p_V))$$

subject to

$$p_W = p_{m_U, \sigma_U} \oplus p_{m_V, \sigma_V} \tag{23}$$

B_W is found as follows.

$$\mu_{B_W}(b_W) = \sup(\mu_{\tilde{P}_W}(p_W)) \tag{24}$$

subject to

$$b_W = \int_{-\infty}^{+\infty} p_W(w) \mu_{A_W}(w) dw \tag{25}$$

The procedures of other operations (subtraction, product, and division) are analogous to those of determination of $W = U + V$.

Computation with Z-numbers is considered below.

Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be continuous Z-numbers describing values of random variables X_1 and X_2. Assume that it is needed to compute $Z_{12} = Z_1 * Z_2, * \in \{+, -, \cdot, /\}$ [3].

At the *first step* we compute $A_{12} = A_1 + A_2$ by using arithmetic of fuzzy numbers. At the *second step* it is needed to compute discretized μ_{p_j}. Construction of continuous μ_{p_j} requires solving a complex variational problem. Discretization of μ_{p_j} leads to reducing of computational complexity. It allows to achieve a required tradeoff between accuracy and computational efficiency. We recall that 'true' pdfs p_1 and p_2 are unknown, and we have to consider all the pdfs p_1 and p_2 satisfying the available restrictions:

$$\sum_{i=1}^{n_j} \mu_{A_j}(x_{ji}) p_j(x_{ji}) \ is \ B_j , \tag{26}$$

At the *third step* we need to compute approximated $\mu_{B_{12}}$. At first, we should compute probability measure of $A_{12} = A_1 * A_2$ given p_{12}, that is, compute probability measure $P(A_{12})$ of the fuzzy event X_{12} *is* A_{12} as

$$P(A_{12}) = \sum_{i=1}^{n} \mu_{A_{12}}(x_{12,i}) p_{12s}(x_{12,i}) \tag{27}$$

Thus, when p_{12} is known, $P(A_{12})$ is a number $P(A_{12}) = b_{12}$. However, what is only known is a fuzzy restriction on pdfs p_{12s} described by the membership function $\mu_{p_{12}}$. Therefore, $P(A_{12})$ will be a fuzzy set B_{12}.

The third chapter deals with the eigensolutions of preferences matrices of Z-Numbers. Eigenvalues and eigenvectors are widely used in various applications. Particularly, these concepts underlie analysis of consistency of a decision maker's (DM) preference knowledge. In real-world problems, DM's knowledge is inherently associated with imprecision and partial reliability. This involves combination of fuzzy and probabilistic information. The concept of a Z-number is a formal

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[3] Aliev R.A., Huseynov O.H., Aliyev R.R., Alizadeh A.V. The Arithmetic of Z-numbers. Theory and Applications, World Scientific, Singapore (2015).

construct to describe such kind of information. In this chapter, we formulate the concepts of Z-number-valued eigenvalue and eigenvector for matrices components of which are Z-numbers. A formal statement of the problem and a solution method for computation of Z-number valued eigenssolutions are proposed.

Nowadays, no studies on matrices with Z-valued information exist. We proceed with research of matrices whose elements are Z-numbers. This implies that we consider an evident synergy of fuzzy and probabilistic uncertainties, which cannot be treated separately. We formulate definitions of Z-matrices, Z-valued eigenvalue and Z-valued eigenvector of a Z-matrix. The problems of computation of Z-valued eigenvalues and Z-valued eigenvectors are formulated, and the solutions are proposed. Construction of Z-valued eigen-solutions, as compared to that of fuzzy analogs, allows to account for partial reliability of information in practical problems. Particularly, given preferences over decision criteria, we can account for fuzziness and partial reliability of resulted criteria importance weights. So far, to the best of our knowledge, the formulation of the problem and its solution have not been tackled.

Let us formulate definitions of eigenvalue and eigenvector of (Z_{ij}).

Z-valued eigenvalue of Z-valued square matrix. A Z-valued eigenvalue of (Z_{ij}) is such a Z-number $Z_\lambda = (A_\lambda, B_\lambda)$ that the following holds:

$$\det(Z_{ij} - Z_\lambda I) = \det\begin{pmatrix} Z_{11} - Z_\lambda & \dots & Z_{1n} \\ . & \dots & . \\ Z_{n1} & \dots & Z_{nn} - Z_\lambda \end{pmatrix} = Z(0) \qquad (28)$$

where I is a traditional (non-fuzzy) identity matrix, $Z(0) = (\tilde{0}, \tilde{1})$, $\tilde{0}, \tilde{1}$ are fuzzy zero and fuzzy unity respectively.

Thus, Z-valued eigenvalue $Z_\lambda = (A_\lambda, B_\lambda)$ is a root of an *n*-th order characteristic equation:

$$Z_0 Z_\lambda{}^n + Z_1 Z_\lambda{}^{n-1} + Z_2 Z_\lambda{}^{n-2} \dots + Z_{n-1} Z_\lambda + Z_n = Z(0), \qquad (29)$$

where $Z_r,\ r=1,...,n$ are coefficients induced by the elements of the Z-valued matrix (Z_{ij}). So, n Z-valued eigenvalues $Z_{\lambda s}=(A_{\lambda s},B_{\lambda s}), s=1,...,n$ are to be found for a Z-valued square matrix (Z_{ij}).

Z-eigenvector of Z-valued matrix. A vector of Z-numbers $(Z_{Yj})=(Z_{Y1}=(A_{Y1},B_{Y1}),...,Z_{Yn}=(A_{Yn},B_{Yn}))^T$ is referred to as a Z-valued eigenvector of Z-valued square matrix (Z_{ij}) if it satisfies the following Z-valued system of linear equations:

$$(Z_{ij})(Z_{Yj})=Z_\lambda(Z_{Yj}) \tag{30}$$

where $Z_\lambda=(A_\lambda,B_\lambda)$ is Z-valued eigenvalue.

Thus, n Z-valued eigenvectors $(Z_{Ysj}), s=1,...,n, j=1,...,n$ are to be found, one for each Z-valued eigenvalue $Z_{\lambda s}=(A_{\lambda s},B_{\lambda s})$.

Statement of the problem

We consider a problem of computation of Z-valued eigenvalues and Z-valued eigenvectors. The generic problem of computation of Z-valued eigenvalues is as follows:

Find $Z_{\lambda s}=(A_{\lambda s},B_{\lambda s}),\ s=1,...,n$ such that

$$\det\begin{pmatrix} Z_{11}-Z_\lambda & ... & Z_{1n} \\ . & ... & . \\ Z_{n1} & ... & Z_{nn}-Z_\lambda \end{pmatrix}=Z(0) \tag{31}$$

Note that in $Z_{ij}=(A_{ij},B_{ij}),\ A_{ij}$ is a fuzzy number and B_{ij} is a fuzzy constraint on probability measure of A_{ij}: $P(A_{ij})=\int_R \mu_{A_{ij}}(x)p_{ij}(x)dx$ is B_{ij}.

This restriction implies that a fuzzy set of possible pdfs p_{ij} exists being induced by B_{ij}. Thus, given $Z_{ij}=(A_{ij},B_{ij}), i,j=1,...,n$ we have to compute $A_{\lambda s}$ and construct fuzzy restriction $B_{\lambda s}$

$$P(A_{\lambda s})=\int_R \mu_{A_{\lambda s}}(x)p_{\lambda s}(x)dx \text{ is } B_{\lambda s}. \tag{32}$$

The problem of computation of $Z^+_{\lambda s}=(A_{\lambda s,}\,p_{\lambda s})$ can be considered as follows.

Find $Z^{+}_{\lambda s} = (A_{\lambda s,} p_{\lambda s})$ such that

$$\det\begin{pmatrix} Z^{+}_{11} - Z^{+}_{\lambda s} & \dots & Z^{+}_{1n} \\ . & \dots & . \\ Z^{+}_{n1} & \dots & Z^{+}_{nn} - Z^{+}_{\lambda s} \end{pmatrix} = Z^{+}(0) \tag{33}$$

where $Z^{+}(0) = (\tilde{0}, p)$. This problem relies on the problems of computation of $A_{\lambda s}$ and $p_{\lambda s}$ described below.

Given fuzzy numbers A_{ij} find $A_{\lambda s}$ such that

$$\det\begin{pmatrix} A_{11} - A_{\lambda s} & \dots & A_{1n} \\ . & \dots & . \\ A_{n1} & \dots & A_{nn} - A_{\lambda s} \end{pmatrix} = \tilde{0}\,. \tag{34}$$

Given pdfs p_{ij} of random variables X_{ij} and the constraint.

$$\det\begin{pmatrix} X_{11} - X_{\lambda s} & \dots & X_{1n} \\ . & \dots & . \\ X_{n1} & \dots & X_{nn} - X_{\lambda s} \end{pmatrix} = 0\,, \tag{35}$$

Find pdf $p_{\lambda s}$ of a desired random variable $X_{\lambda s}$. Symbolically, we can describe this problem as:

Find $p_{\lambda s}$. such that

$$\det\begin{pmatrix} p_{11} - p_{\lambda s} & \dots & p_{1n} \\ . & \dots & . \\ p_{n1} & \dots & p_{nn} - p_{\lambda s} \end{pmatrix} = 0. \tag{36}$$

Next, we have to find Z-valued eigenvectors (Z_{Ysj}) as solutions of a Z-valued linear system (30). (Z_{Ysj}) can be found through computation of $(Z^{+}_{Ysj} = (A_{Ysj}, p_{Ysj}))$ such that vectors (A_{Ysj}) and (p_{Ysj}) are solutions of two problems associated with system (30).

Given $A_{\lambda s}$, find (A_{Ysj}) that satisfies the following linear system:

$$\begin{pmatrix} A_{11} & \dots & A_{1n} \\ . & \dots & . \\ A_{n1} & \dots & A_{nn} \end{pmatrix}\begin{pmatrix} A_{Ys1} \\ \dots \\ A_{Ysn} \end{pmatrix} = A_{\lambda s}\begin{pmatrix} A_{Ys1} \\ \dots \\ A_{Ysn} \end{pmatrix} \tag{37}$$

Given random variable $X_{\lambda s}$ compute the random vector (Y_{sj}) by solving the following linear system:

$$\begin{pmatrix} X_{11} & \dots & X_{1n} \\ . & \dots & . \\ X_{n1} & \dots & X_{nn} \end{pmatrix} \begin{pmatrix} Y_{s1} \\ \dots \\ Y_{sn} \end{pmatrix} = X_{\lambda s} \begin{pmatrix} Y_{s1} \\ \dots \\ Y_{sn} \end{pmatrix} \quad (38)$$

Symbolically, the last problem can be formulated as given below. Find the vector of pdfs $(p_{Ysj}),\ s = 1,...,n$ that satisfies the following system:

$$\begin{pmatrix} p_{11} & \dots & p_{1n} \\ . & \dots & . \\ p_{n1} & \dots & p_{nn} \end{pmatrix} \begin{pmatrix} p_{Ys1} \\ \dots \\ p_{Ysn} \end{pmatrix} = p_{\lambda s} \begin{pmatrix} p_{Ys1} \\ \dots \\ p_{Ysn} \end{pmatrix}. \quad (39)$$

Solution to the problem

At the first step, we have to consider a Z^+-matrix $(Z_{ij}^+ = (A_{ij}, p_{ij}))$ related to Z-matrix $(Z_{ij} = (A_{ij}, B_{ij}))$. For simplicity, pdf p_{ij} is considered to be normal $p_{ij} = (m_{ij}, \sigma_{ij})$. Formally, given $b_{ij} \in B_{ij}$, we have to find σ_{ij} by solving the following optimization problem:

$$\sum_{k=1}^{N} \mu_{A_{ij}}(x_{ijk}) \frac{1}{\sqrt{2\pi\sigma_{ij}}} e^{-\frac{(m_{ij} - x_{ijk})^2}{\sigma_{ij}}} \Delta x \to b_{ij} \quad (40)$$

s.t.

$$m_{ij} = \sum_{k=1}^{N} \frac{\mu_{A_{ij}}(x_{ijk}) x_{ijk}}{\mu_{A_{ij}}(x_{ijk})} \quad (41)$$

(41) is compatibility constraint[1].

[1] Zadeh L.A. A note on Z-numbers. Information Sciences, 181(14), pp. 2923–2932 (2011).

Given fuzzy matrix (A_{ij}) and the related random matrix (p_{ij}), we compute $Z^{+}_{\lambda s} = (A_{\lambda s}, p_{\lambda s})$, where $A_{\lambda s}$ is a fuzzy eigenvalue of (A_{ij}), and $p_{\lambda s}$ is a probabilistic eigenvalue of (p_{ij}). $A_{\lambda s}$ is computed by using a decomposition approach[7].

Given n eigenvalues $Z^{+}_{\lambda s} = (A_{\lambda s}, p_{\lambda s})$, we have to compute the corresponding eigenvectors $(Z^{+}_{Ysj} = (A_{Ysj}, p_{Ysj})), s = 1,...,n$.

A fuzzy eigenvector with triangular fuzzy numbers (TFNs)-based components $(A_{Ysj} = (a_{Ysjl}, a_{Ysjm}, a_{Ysju}))$ is computed on the basis of the decomposition approach.

Example

Consider the following Z-valued matrix, where components of Z-numbers are TFNs:

$$(Z_{ij}) = \begin{pmatrix} ((0.93, 0.95, 1), (0.89, 0.95, 1)) & ((2, 2.5, 3), (0.7, 0.8, 1)) \\ ((0.33, 0.4, 0.5), (0.7, 0.8, 1)) & ((0.93, 0.93, 0.95), (0.89, 0.95, 1)) \end{pmatrix}$$

The computed fuzzy and probabilistic eigenvalues are $A_{\lambda 1} = (1.7, 1.95, 2.2)$, $A_{\lambda 2} = (-0.23, -0.04, 0.8)$ and $p_{\lambda 1} = (1.94, 0.0365), p_{\lambda 2} = (-0.054, 0.025)$. Thus, Z-valued eigenvector (Z_{Y1j}) is

$$(Z_{Y1j}) = \begin{pmatrix} ((0.929, 0.932, 0.933), (0, 0.22, 0.3)) \\ ((0.359, 0.36, 0.371), (0.13, 0.22, 0.22)) \end{pmatrix}.$$

Consistency-Driven Preferences

Decision making is based on preferences over alternatives and choice criteria. A DM's preference may be formally described by a pairwise comparison matrix (PCM) (a_{ij}), where an a_{ij} denotes a degree

[7] Praščević N. and Praščević Ž., Application of fuzzy AHP method based on eigenvalues for decision making in construction industry, Technical Gazette 23(1) (2016) 57–64.

to which an *i-th* alternative (criterion) is preferred to *j*-th one[8]. Natural conditions used for a_{ij} are $a_{ii} = 1$ and $a_{ji} = 1/a_{ij}$ (reciprocity), $\forall i, j = 1,...,n$. Traditionally, consistency of (a_{ij}) is based on multiplicative transitivity condition (though different constructs are also used):

$$a_{ij} a_{jk} = a_{ik}, \forall i, j, k. \quad (42)$$

This implies that degree of preference a_{ik} is equal to the product of preferences degrees staying at all possible ways from *i* to *k* through *j*. Fulfillment of this condition is often problematic due to low computational abilities of a human brain. An inconsistency is related to violation of this multiplicative transitivity condition.

Being imprecise and partially consistent, real-world preferences are also partially reliable. The reasons are restricted competence of DM's, complexity of alternatives, imperfect decision-relevant information, psychological biases etc. Up to day, no works have been proposed on consistency of partially reliable preferences.

In this section we propose a new approach for construction of consistency-driven partially reliable preferences described by PCM with Z-number-valued entries[9].

Z-valued PCM. A Z-valued PCM (Z_{ij}) is a square matrix of Z-numbers:

$$(Z_{ij} = (A_{ij}, B_{ij})) = \begin{pmatrix} Z_{11} = (A_{11}, B_{11}) & \dots & Z_{1n} = (A_{1n}, B_{1n}) \\ . & \dots & . \\ Z_{n1} = (A_{n1}, B_{n1}) & \dots & Z_{nn} = (A_{nn}, B_{nn}) \end{pmatrix}. \quad (43)$$

A Z-number $Z_{ij} = (A_{ij}, B_{ij}),\ i, j = 1,...,n$ describes partially reliable information on degree of preference for *i-th* alternative (criterion) against *j*-th one.

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[8] Brunelli M. A survey of inconsistency indices for pairwise comparisons, International Journal of General Systems, 47(8), pp. 751-771 (2018).

[9] Aliev R.A., Guirimov B.G., Huseynov O.H., Aliyev R.R. A consistency-driven approach to construction of Z-number-valued pairwise comparison matrices. Iranian Journal of Fuzzy Systems, 18(4), pp. 37-49.

An inconsistency index for Z-number-valued PCM. An inconsistency index K for Z-number-valued PCM (Z_{ij}) is defined as follows[10]:

$$K((Z_{ij})) = \max_{i<j<k} \min \left\{ D\left((1,1), \left(\frac{Z_{ik}}{Z_{ij}Z_{jk}}\right)\right) D\left((1,1), \left(\frac{Z_{ij}Z_{jk}}{Z_{ik}}\right)\right) \right\}, \quad (44)$$

Let us consider a problem of generation of consistent PCM (Z'_{ij}) most similar to a given inconsistent PCM (Z_{ij}). The elements of inconsistent Z-matrix (Z_{ij}) will be considered as a perturbation of the elements of matrix (Z'_{ij}) for which reciprocity and consistency are verified. We have to change elements of (Z_{ij}) in order to arrive at (Z'_{ij}). The problem is formulated as follows.

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} D(Z_{ij}, Z'_{ij}) \to \min \quad (45)$$

s.t. multiplicative reciprocity:

$$Z'_{ij} Z'_{ji} = Z(1) \ , \quad (46)$$

multiplicative transitivity:

$$Z'_{ij} Z'_{jk} = Z'_{ik} \ , \quad (47)$$

non-negativity:

$$Z'_{ij} \geq Z(0) \ , \ i,j=1,\ldots,n \quad (48)$$

Problem (45)-(48) is a non-linear optimization problem characterized by fuzzy and probabilistic uncertainties. Taking into account these features, it is needed to develop a solution approach relying on differential evolution (DE) optimization technique[10].

Example. Preferences over criteria in a market selection problem.

Let us consider extraction of a consistent Z-valued matrix to describe preferences over multiple criteria in a foreign market

[10] Storn R., Price K. Differential Evolution - A simple and efficient adaptive scheme for global optimization over continuous spaces. J Global Optim, 11, pp. 341-359 (1997).

selection problem[11]. We will deal with three criteria that describe a series of economical and institutional characteristics: Institutional Proximity (C_1) (government performance and economic freedom issues), Economic Proximity (C_2) (socioeconomic issues) and Social and Cultural Proximity (C_3) (cultural characteristics).

Information can be formalized by a 3×3 matrix of Z-numbers with TFNs-based components:

$$\begin{pmatrix} Z_{11}=((0.93,0.95,0.97),(0.95,0.98,1)) & Z_{12}=((0.327,0.333,0.34),(0.7,0.8,0.9)) & Z_{13}=((3.92,4,4.08),(0.7,0.8,0.9)) \\ Z_{21}=((2.94,3,3.06),(0.7,0.8,0.9)) & Z_{22}=((0.93,0.95,0.97),(0.95,0.98,1)) & Z_{23}=((8.82,9,9.02),(0.6,0.7,0.8)) \\ Z_{31}=((0.245,0.25,0.255),(0.7,0.8,0.9)) & Z_{32}=((0.1108,0.111,0.113),(0.6,0.7,0.8)) & Z_{33}=((0.93,0.95,0.97),(0.95,0.98,1)) \end{pmatrix}$$

Z_{ij} denotes a Z-valued degree to which the *i*-th criterion is preferred to the *j*-th one. For example, $Z_{21}=((2.94,3,3.06),(0.7,0.8,0.9))$ is a Z-valued degree to which C_2 is preferred to C_1. The value of computed inconsistency index K for the considered matrix (Z_{ij}) is $K((Z_{ij}))=0.31$. Let us consider extraction of consistent Z-valued matrix (Z'_{ij}) closest to the initial one. For this purpose, optimization problem (45)-(48) will be solved by using the proposed method.

Optimal Z-valued matrix $(Z'_{ij})=MP(V'_{best})$ is retrieved:

[11] Aliyev, R.R. Construction of consistent Z-preferences in decision making for a foreign market selection // Advances in Intelligent Systems and Computing, Springer, vol. 1306, p. 30-37 (2021)

$$\left(\begin{array}{c} Z_{11}=((1.000761, 1.002326, 1.002326),(0.9996, 0.9998, 0.9998)) \\ Z_{21}=((2.482609, 2.482835, 2.482835),(0.707104, 0.707104, 0.991447)) \\ Z_{31}=((0.273967,\ 0.273967,\ 0.273967),(0.007870,\ 0.008223,\ 0.502359)) \end{array}\right.$$

$$\begin{array}{c} Z_{12}=((0.402933,\ 0.402933,\ 0.402935),(0.72, 0.995931,\ 0.995931)) \\ Z_{22}=((0.995452, 1, 1),(0.95, 0.98, 1)) \\ Z_{32}=((0.110189,\ 0.110424,\ 0.110424),(0.499450,\ 0.999329,\ 0.999329)) \end{array}$$

$$\left.\begin{array}{c} Z_{13}=((3.650363,\ 3.651220,\ 3.651235),(0.996759,\ 0.997332,\ 0.997332)) \\ Z_{23}=((9.060410,\ 9.060411,\ 9.064086),(0.994743, 0.994743, 0.995316)) \\ Z_{33}=((0.998922,\ 1.002763,\ 1.002763),(0.99867, 0.99986, 0.999886)) \end{array}\right)$$

At the final step, we have to verify whether the value of K for the obtained (Z'_{ij}) exceeds a predefined threshold $\theta_K = 0.1$. The computed value of K is $K((Z'_{ij})) = 0.003$ which does not exceed θ_K. Thus, the obtained matrix can be considered as consistent.

In the fourth chapter we consider Z-relation based decision making method. Conceptually, fuzzy relation equations are used to formalize imprecise information about dependence of variables of interest sourced from human knowledge. However, real-world information is also characterized by partial reliability of the sources.

Nowadays, no works on Z-relations and Z-relation equations exist. In this chapter, we propose the concept of Z-valued relation that serves for modeling of dependence between sets under fuzziness and partial reliability. Main operations over Z-relations are formulated. Further one type of Z-relation equations is formulated.

On this basis, an approach to decision making under Z-valued information is proposed.

Z-relation. A Z-valued relation R between X and Y is represented as

$$R = \{(x_i, y_j), R(x_i, y_j)\}, \tag{49}$$

where $R(x_i, y_j)$ is a Z-number $R(x_i, y_j) = Z_{Rij}(A_{Rij}, B_{ij})$.

That is, R is a matrix $(Z_{R\,ij})$:

$$\left(Z_{R\,ij}\right) = \begin{pmatrix} Z_{R\,11} & \dots & Z_{R\,1n} \\ \dots & \dots & \dots \\ Z_{R\,n1} & \dots & Z_{R\,nn} \end{pmatrix} \tag{50}$$

An element $Z_{R\,ij}\left(A_{R\,ij}, B_{R\,ij}\right), A_{R\,ij} \subset [0,1]$ describe Z-valued degree of relation between x_i and y_j.

Z-relation equation

We consider Z-relation equation of the following form:

$$(Z_{R\,jk}) \circ (Z_{Qij}) = (Z_{T\,ik}), \tag{51}$$

where $(Z_{Qij}),(Z_{R\,jk}),(Z_{T\,ik})$ are Z-relations and $\circ$ denotes *max-min* composition.

Solving of the Z-relation equation implies determination of one of Z-relations $(Z_{Qij}),(Z_{Rij})$ given another one and the result of composition, $(Z_{T\,ij})$:

Given (Z_{Qij}) *and* $(Z_{T\,ik})$ *determine* $(Z_{R\,jk})$

or

Given $(Z_{R\,jk})$ *and* $(Z_{T\,ik})$ *determine* (Z_{Qij})

Let us consider a decision problem under Z-valued information formalized by Z-valued relations (Table 1):

Table 1.

A decision table

	a_1	…	a_M
C_1	$(Z^{value}_{(a_1,C_1)ij})$	…	$(Z^{value}_{(a_M,C_1)ij})$
…	…	…	…
C_N	$(Z^{value}_{(a_1,C_N)ij})$	…	$(Z^{value}_{(a_M,C_N)ij})$

In Table 1, $a_1,...,a_M$ denote alternatives, $C_1,...,C_N$ denote decision criteria, $(Z^{value}_{(a_1,C_1)ij}),...,(Z^{value}_{(a_M,C_N)ij})$ denote one-dimensional criteria evaluations of alternatives. Also, an importance degree (e.g. importance weight) described by one-dimensional Z-relation $(Z^{w}_{C\,ij})$ is associated with each criterion $C \in \{C_1,...,C_N\}$. The decision problem requires to determine the best alternative:

Find $a^* \in \{a_1,...,a_M\}$ such that $a^* \succsim a, \forall a \in \{a_1,...,a_M\}$, where $\succsim$ denotes preference.

At the first stage, we need to find two-dimensional Z-relations $(Z^{value}_{(a,C)ij})$, $a \in \{a_1,...,a_M\}, C \in \{C_1,...,C_N\}$ between criteria evaluation $(Z^{value}_{(a,C)ij})$ with respect to a criterion C and the criteria importance (Z^{w}_{Cij}) by solving Z-relation equation of the form:

$$(Z_{(a,C)ij}) \circ (Z^{value}_{(a,C)ij}) = (Z^{w}_{Cij}) \tag{52}$$

$(Z_{(a,C)ij})$ is found as a maximal solution $(Z_{(a,C)'ij})$:

$$(Z_{(a,C)'ij}) = (Z^{value}_{(a,C)ij})^{-1} \alpha (Z^{w}_{Cij}) . \tag{53}$$

At the second stage, we need to find evaluation (Z_{aij}) of each alternative with respect to all the criteria as an intersection:

$$(Z_{aij}) = (Z_{(a,C_1)'ij}) \cap ... \cap (Z_{(a,C_N)'ij}),\ a \in \{a_1,...,a_M\} . \tag{54}$$

At the third stage we have to compute overall evaluations of alternatives (V_a), $a \in \{a_1,...,a_M\}$. Assume that significance of each criteria is evaluated as 'indeed critical, sure'. We will compute overall evaluations (V_a) as $\hat{\alpha}$-composition:

$$(V_a) = ((Z_{aij}) \hat{\alpha} ((Z^{w}_{Cij})^*)^{-1})^{-1},\ a \in \{a_1,...,a_M\} , \tag{55}$$

where $(Z^{w}_{Cij})^*$ is the highest Z-valued importance degree (of criteria).

Finally, at the fourth stage, the best alternative is found as one with the highest overall evaluation:

Find $a^* \in \{a_1,...,a_M\}$ such that $(V_{a^*}) \geq (V_a), a \in \{a_1,...,a_M\}$.

In the fifth chapter we suggest two multi-attribute decision making tools based on Z-number concept. We investigate MADM problem, where the attribute values are Z-numbers, and the weight information on attributes are partially reliable. The presented method is based on overall criteria positive ideal and negative ideal solution of alternatives and distance between Z-vectors. Final decision alternative is selected on basis of degree of membership of candidates belonging to the positive ideal solution.

Assume that $A=\{A_1, A_2, ..., A_n\}$ is a set of alternatives and $C=\{C_1, C_2, ..., C_m\}$ is a set of attributes. Every attribute $C_j, j=\overline{1,m}$ is characterized by weight W_j assigned by expert or decision maker. As we deal with Z-information valued decision environment, the characteristic of the alternative $A_i, i=\overline{1,n}$ on attribute $C_j\left(j=\overline{1,m}\right)$ is described in the form

$$A_i=\{(Z(A_{i1}, B_{i1}), Z(A_{i2}, B_{i2}), ..., Z(A_{ij}, B_{ij}), Z(A_{im}, B_{im})\} \quad (56)$$

where $Z(A_{ij}, B_{ij})$ is evaluation of an alternative A_i with respect to an attribute C_j. Value of attributes and weights of attributes are usually derived from experts and are vague and characterized with partial reliability. In this case, the weights $W_j, i=\overline{1,m}$ are represented as

$$W_j=\{Z(A_j^w, B_j^w)\},\ j=\overline{1,m} \quad (57)$$

where A_j^w is value of weight of j-th is attribute, B_j^w is reliability of this value. Hence, we can represent decision matrix $D_{n\times m}$ as in Table 2.

Table 2.

Decision Z-matrix

		C_1	C_2	...	C_m
	A_1	$[Z(A_{11}, B_{11})]$	$[Z(A_{12}, B_{12})]$	...	$[Z(A_{1m}, B_{1m})]$
$D_{n\times m}=$	A_1	$[Z(A_{21}, B_{21})]$	$[Z(A_{22}, B_{22})]$	...	$[Z(A_{2m}, B_{2m})]$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	A_n	$[Z(A_{n1}, B_{n1})]$	$[Z(A_{n2}, B_{n2})]$	...	$[Z(A_{nm}, B_{nm})]$

In this case we will use the concept of positive and negative ideal points in MADM[12]. We present a positive ideal Z-point for attributes as

$$A_p^{id}=\left(Z\left(A_{p1}^{id}, B_{p1}^{id}\right), Z\left(A_{p2}^{id}, B_{p2}^{id}\right), ..., Z\left(A_{pm}^{id}, B_{pm}^{id}\right)\right) \quad (58)$$

A negative ideal point will be discribed as

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[12] Hwang C.L., Yoon K. Multiple attribute decision making methods and applications. Springer: Berlin Heidelberg (1981).

$$A_N^{id} = \left(Z\left(A_{N_1}^{id}, B_{N_1}^{id}\right), Z\left(A_{N_2}^{id}, B_{N_2}^{id}\right), ..., Z\left(A_{N_m}^{id}, B_{N_m}^{id}\right)\right) \quad (59)$$

Solution of the stated decision making problem, i.e. choice of the best alternative among $A = \{A_1, A_2, ..., A_n\}$ consist of the following steps:

1. Weighted distances d_{ip} between i -th alternative and positive ideal solutions (58) is defined by (7).
2. Weighted distances d_{iN} between i -th alternative and negative ideal solutions (59) is defined by (7).
3. Degree of membership $r_i, i = \overline{1,n}$, of each alternatives belonging to the positive ideal solution is calculated[13]:

$$r_i = \frac{1}{1 + \left(\dfrac{d_{ip}}{d_{iN}}\right)} \quad (60)$$

4. Final decision alternative is selected as $\max(r_i), i = 1, n$

In this chapter we also suggest similarity-based decision method for the considered problem. We apply Jaccard similarity measure between an alternative A_i and ideal alternative A^{id} (58). Jaccard similarity measure between Z-number valued vectors is represented as

$$J_{s_i} = \sum_{j=1}^{m} \left\{Z\left(A_j^w, B_j^w\right)\right\} \times \left(\frac{1}{2} \frac{\sum_{k=1}^{K} \mu_{A_i}(x_k) \cdot \mu_{A^{id}}(x_k)}{\sum_{k=1}^{K}\left(\mu_{A_i}(x_k)\right)^2 + \sum_{k=1}^{K}\left(\mu_{A^{id}}(x_k)\right)^2 - \sum_{k=1}^{K} \mu_{A_i}(x_k) \cdot \mu_{A^{id}}(x_k)} + \right.$$
$$\left. + \frac{1}{2} \frac{\sum_{k=1}^{K} \mu_{B_i}(x_k) \cdot \mu_{B^{id}}(x_k)}{\sum_{k=1}^{K}\left(\mu_{B_i}(x_k)\right)^2 + \sum_{k=1}^{K}\left(\mu_{B^{id}}(x_k)\right)^2 - \sum_{k=1}^{K} \mu_{B_i}(x_k) \cdot \mu_{B^{id}}(x_k)} \right) \quad (61)$$

After calculation of the similarity measure J_{s_i} between alternatives A_i, $i = \overline{1,n}$ and the ideal alternative A^{id} using (58) on basis of ranking we can determine the best alternative.

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[13] Tong H., Zhang S. A fuzzy multi-attribute decision making algorithm for web services selection based on QoS. IEEE Asia-Pacific Conference on Services Computing, pp. 51-57 (2006).

In the sixth chapter we give application of suggested in this dissertation decision making methods to different real-life problems.+

Country Selection for Business Location by using consistency-driven Z-preferences

Country selection for doing business is based on rising globalized environment and intensity of competition and escalated dependency of firms for internationalized business activities in order to secure their survival in the world market. Comprehensive improvement of technological innovations, reduced barriers and favorable government incentives encouraged companies to internationally expand and maximize their sustainable growth.

Statement of the problem

Let us consider a problem of decision making in Foreign Market Entry (FME) problem. Four decision criteria that describe economical and institutional characteristics of countries are considered: Institutional Proximity, C_1, Economic Proximity, C_2, Social and Cultural Proximity, C_3, and Structural Competition Proximity C_4. Criterion C_1 represents governance performance and economic freedom. Criterion C_2 describes both domestic development (such as socioeconomic progress, household's standard of living) and global competitiveness issues. Criterion C_3 concerns cultural characteristics (an extent to which collectivism or individualism-based behavior is intrinsic to a country, issues related to communication with customers etc.). C_4 is a complex criterion that expresses such indicators as Regulatory Environment Attractiveness, Market, Resources and Human Capital Attractiveness. Particularly, it concerns paying taxes, R&D, Information Communication Technologies (ICT) factors[14].

[14] Hortacsu A, Tektas A. Modeling the Country Selection Decision in Retail Internationalization, International Journal of Economics and Management Engineering, 3(7), pp. 1068-1075 (2009).

The mentioned criteria are used to evaluate nine alternatives (countries): $a_1,...,a_9$. Due to high level of uncertainty in the considered problem, the decision-relevant information on criteria evaluations of alternatives and preferences over criteria is characterized by fuzziness and partial reliability. In view of this, we use partially reliable preference degrees of the Saaty scale to represent comparative importance of criteria (Table 3):

Table 3.

Z-number-valued preference knowledge about criteria importance

	C_1	C_2	C_3	C_4
C_1	(equally important, very sure)	(moderately to strongly less important, sure)	(strongly more important, sure)	(moderately less important, sure)
C_2	(moderately to strongly more important, sure)	(equally important, very sure)	(strongly more important, almost sure)	(moderately less important, sure)
C_3	(strongly less important, sure)	(strongly less important, almost sure)	(equally important, very sure)	(strongly less important, rather sure)
C_4	(moderately more important, sure)	(moderately more important, sure)	(strongly more important, rather sure)	(equally important, very sure)

This information can be formalized by a 4×4 PCM of Z-numbers with triangular fuzzy numbers (TFNs)-based components:

$$\begin{pmatrix} Z_{11}=((0.99,1,1),(0.99,1,1)) & Z_{12}=((0.245,0.25,0.255),(0.7,0.8,0.9)) & Z_{13}=((4.9,5,5.1),(0.7,0.8,0.9)) & Z_{14}=((0.327,0.333,0.34),(0.7,0.8,0.9)) \\ Z_{21}=((3.92,4,4.08),(0.7,0.8,0.9)) & Z_{22}=((0.99,1,1),(0.99,1,1)) & Z_{23}=((4.9,5,5.1),(0.6,0.7,0.8)) & Z_{24}=((0.327,0.333,0.34),(0.7,0.8,0.9)) \\ Z_{31}=((0.196,0.2,0.204),(0.7,0.8,0.9)) & Z_{32}=((0.196,0.2,0.204),(0.6,0.7,0.8)) & Z_{33}=((0.99,1,1),(0.99,1,1)) & Z_{34}=((0.196,0.2,0.204),(0.5,0.6,0.7)) \\ Z_{41}=((2.94,3,3.06),(0.7,0.8,0.9)) & Z_{42}=((2.94,3,3.06),(0.7,0.8,0.9)) & Z_{43}=((4.9,5,5.1),(0.5,0.6,0.7)) & Z_{44}=((0.99,1,1),(0.99,1,1)) \end{pmatrix} \quad (62)$$

Z_{ij} denotes a Z-number-valued degree to which the i-th criterion is preferred to the j-th one. For example, $Z_{21}=((3.92,4,4.08),(0.7,0.8,0.9))$ is a Z-number-valued degree to which C_2 is preferred to C_1.

Solution of the problem

The solution method includes several stages. At the 1st stage we need to generate consistent PCM (Z'_{ij}) closest to the given matrix (Z_{ij}) (62) by using the method proposed by the author in chapter 3. The problem is formulated as follows.

The objective function is formalized in accordance with (45). The use of (45) implies that one needs to minimize distance between elements of matrix (Z'_{ij}) and those of initial inconsistent matrix (Z_{ij}) subject to (46)-(48).

In (46) $Z(1)=(A,B)$ is such that A,B are fuzzy singletons $A=1,\ B=1.$ These are necessary conditions implying that if i-th criterion is more important than j-th one to a Z-number-valued degree Z'_{ij}, then the latter is less important than i-th one to the same Z-number-valued degree. Multiplicative transitivity constraints (47) imply that if i-th criterion is Z'_{ij} times more important than j-th one, and the latter is Z'_{jk} times more important than k-th one, then preference of i-th criterion against k-th one should be $Z'_{ik}=Z'_{ij}Z'_{jk}$.

Next constraints imply that preference degrees are non-negative, where $Z(0)=(A,B)$ is such that A,B are fuzzy singletons $A=0,\ B=1$.

Optimal Z-number-valued matrix (Z'_{ij}) is retrieved as:

$$\begin{pmatrix} Z_{11}=((1,1,1),(1,1,1)) & Z_{12}=(0.476,0.476,0.476),(0.92,1,1)) & Z_{13}=((1.6,1.6,1.6),(0.36,0.5,0.5)) & Z_{14}=((0.52,0.52,0.52),(0.68,1,1)) \\ Z_{21}=((2.13,2.13,2.13),(0.76,1,1)) & Z_{22}=((1,1,1),(1,1,1)) & Z_{23}=((3.4,3.4,3.4),(0.26,0.5,0.5)) & Z_{24}=((1.1,1.1,1.1),(0.5,1,1)) \\ Z_{31}=((0.196,0.2,0.204),(0.7,0.8,0.9)) & Z_{32}=((0.3,0.3,0.3),(1,1,1)) & Z_{33}=((1,1,1),(1,1,1)) & Z_{34}=((0.32,0.32,0.33),(0.69,1,1)) \\ Z_{41}=((1.97,1.97,1.97),(0.47,0.5,0.5)) & Z_{42}=((0.92,0.92,0.92),(0.47,0.5,0.5)) & Z_{43}=((3.12,3.13,3.13),(0.42,0.5,0.5)) & Z_{44}=((1,1,1),(1,1,1)) \end{pmatrix}$$

We have to verify whether the value of K for the obtained (Z'_{ij}) exceeds a predefined threshold $\theta_K=0.1$. The computed value of K is $K((Z'_{ij}))=0.08$ which does not exceed θ_K. Thus, the obtained matrix can be considered as consistent.

At the 2nd stage, given the obtained consistent (Z'_{ij}), we need to compute the importance weights of criteria given the eigenvector and the maximal eigenvalue of (Z'_{ij}), calculated by method given in chapter 3.

Thus, the criteria weights are as follows:

$$Z_{w_1}=((0.111228,0.175994,0.315706),(0,1,1)),$$
$$Z_{w_2}=((0.111228,0.371348,0.666315),(0,1,1)),$$
$$Z_{w_3}=((0.111228,0.111228,0.200529),(0,1,1)),$$
$$Z_{w_4}=((0.111228,0.341429,0.612375),(0,1,1)).$$

Now, we need to determine an ideal alternative as one with the highest values of the Z-valued weighted criteria. The obtained ideal alternative $a^*=(Z_1^*,...,Z_4^*)$ is shown in Table 4.

Table 4.

The ideal alternative

	A			B			A			B		
a*	0.02	0.02	0.05	0.00	1.00	1.00	0.01	0.04	0.11	0.00	0.99	0.99
	A			B			A			B		
a*	0.04	0.04	0.07	0.00	1.00	1.00	0.01	0.04	0.09	0.00	1.00	1.00

Next, we need to compute values of distance between each alternative $a_i = (Z_{C_{i1}},...,Z_{C_{i4}})$ and the ideal one $a^* = (Z_1^*,...,Z_4^*)$. The distance is found as follows:

$$D(a_i, a^*) = \sqrt{\sum_{j=1}^{4} D^2(Z_{ij}, Z^*)}$$

The obtained results are given in Table 5.

Table 5.

The values of distance

Alternative	$D(a_i, a^*)$
a_1	0,021942
a_2	0,007835
a_3	0,027806
a_4	0,029652
a_5	0,063197
a_6	0,030289
a_7	0,0027
a_8	0,067073
a_9	0,066013

Finally, at the 4th stage, we rank the alternatives as

$$a_i \succsim a_k \text{ iff } D(a_i, a^*) \le D(a_k, a^*) .$$

Thus, we have the following results:

$$a_7 \succ a_2 \succ a_1 \succ a_3 \succ a_4 \succ a_6 \succ a_5 \succ a_9 \succ a_8 .$$

Z-decision making in hotel business

Let us consider problem of decision making under Z-information in business area. A management of a hotel should make a decision concerning a construction of an additional wing. The alternatives are buildings with 30 **(** f_1 **)**, 40 **(** f_2 **)** and 50 **(** f_3 **)** rooms. The results of each decision depend on a combination of local government legislation and competition in the field. With respect to this, three states of nature are considered: positive legislation and low competition (s_1), positive legislation and strong competition (s_2), no legislation and low competition (s_3). The outcomes (results) of each decision are values of anticipated payoffs (in percentage) described by Z-numbers. The problem is to find how many rooms to build in order to maximize the return on investment.

Z -information for the utilities of each act taken at various states of nature and probabilities of are provided in Table 6 and Table 7 respectively.

Table 6.

The utility values of actions under various states

	s_1	s_2	s_3
f_1	(high; likely)	(below than high; likely)	(medium; likely)
f_2	(below than high; likely)	(low; likely)	(below than high; likely)
f_3	(below than high; likely)	(high; likely)	(medium; likely)

Table 7.

The values of probabilities of states of nature

$P(s_1)$ =(medium; quite sure)	$P(s_2)$ =(more than medium; quite sure)	$P(s_2)$ =(low; quite sure)

Given these data, we get the expected values of utility for acts f_1, f_2, f_3:

$$Z(A_{U_1}, B_{U_1}) = \tilde{Z}_{U(f_1)} = \tilde{Z}_{v_{s_1}(f_1(s_1))} \text{ x } \tilde{Z}_{P(s_1)} + \tilde{Z}_{v_{s_2}(f_1(s_2))} \text{ x } \tilde{Z}_{P(s_2)} + \tilde{Z}_{v_{s_3}(f_1(s_3))} \text{ x } \tilde{Z}_{P(s_3)},$$

$$Z(A_{U_2}, B_{U_2}) = \tilde{Z}_{U(f_2)} = \tilde{Z}_{v_{s_1}(f_2(s_1))} \text{ x } \tilde{Z}_{P(s_1)} + \tilde{Z}_{v_{s_2}(f_2(s_2))} \text{ x } \tilde{Z}_{P(s_2)} + \tilde{Z}_{v_{s_3}(f_2(s_3))} \text{ x } \tilde{Z}_{P(s_3)},$$

$$Z(A_{U_3}, B_{U_3}) = \tilde{Z}_{U(f_3)} = \tilde{Z}_{v_{s_1}(f_3(s_1))} \text{ x } \tilde{Z}_{P(s_1)} + \tilde{Z}_{v_{s_2}(f_3(s_2))} \text{ x } \tilde{Z}_{P(s_2)} + \tilde{Z}_{v_{s_3}(f_3(s_3))} \text{ x } \tilde{Z}_{P(s_3)}.$$

$$Z(A_{U_1}, B_{U_1}) = ((0,1,3),(0.1,0.25,0.28)),$$

$$Z(A_{U_2}, B_{U_2}) = ((0,0.85,3),(0.09,0.27,0.28)),$$

$$Z(A_{U_3}, B_{U_3}) = ((0,0.8,3),(0.1,0.27,0.28)).$$

Membership functions of obtained values are shown in Figs. 1, 2, and 3 respectively.

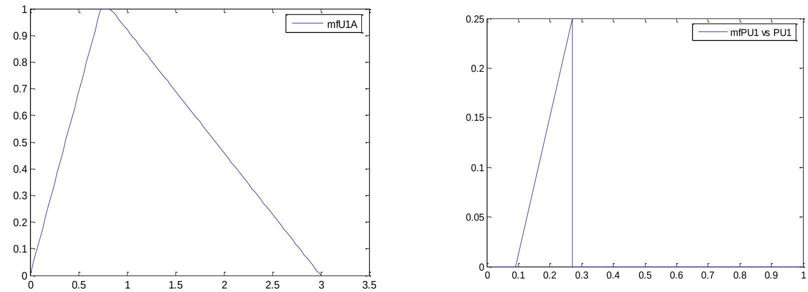

Figure 1. Membership function of A_{U_1} and B_{U_1}

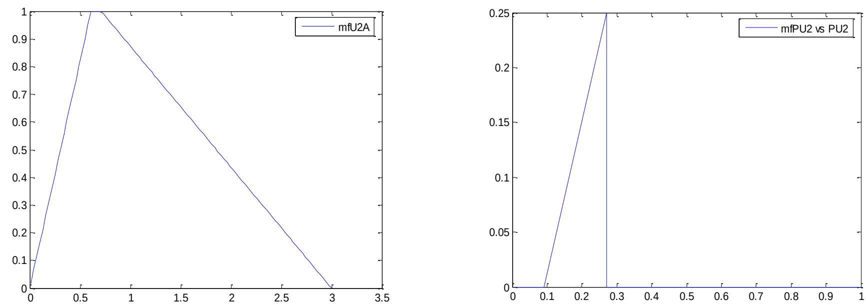

Figure 2. Membership function of A_{U_2} and B_{U_2}

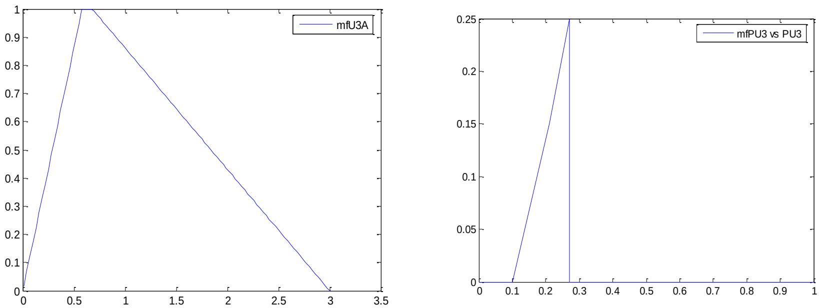

Figure 3. Membership function of A_{U_3} and B_{U_3}

Ranking of fuzzy values of utilities gives a preference to the first alternative, i.e. $f_1 \succ f_3 \succ f_2$.

Z-decision making for Web services selection

We consider MADM for Web services selection problem. Today a wide variety of services are offered that can satisfy quality of services for agents. The number of options, i.e. Web services is 8 A_1, A_2, A_3 … A_8. An agent has to make a decision taking into account 5 attributes C_1 (cost), C_2 (time), C_3 (reliability), C_4 (availability), C_5 (repetition). In this case all 8 alternatives are evaluated under 5 attributes by Z-numbers. Components of these Z-numbers are presented by TFS (normalized decision matrix is shown in Tables 8 and 9).

Table 8.

Decision matrix

	C_1	C_2	C_3
A1	(0.45 0.5 0.55) (0.5 0.6 0.7)	(0.441 0.49 0.539)(0.5 0.6 0.7)	(0.621 0.69 0.759))(0.5 0.6 0.7)
A2	(0.126 0.14 0.154) (0.5 0.6 0.7)	(0.531 0.59 0.649)(0.5 0.6 0.7)	(0.423 0.47 0.517))(0.5 0.6 0.7)
A3	(0.225 0.25 0.275)(0.5 0.6 0.7)	(0.711 0.79 0.869) (0.7 0.8 0.9)	(0.27 0.3 0.33))(0.5 0.6 0.7)
A4	(0.612 0.68 0.748)(0.5 0.6 0.7)	(0.603 0.67 0.737)(0.5 0.6 0.7)	(0.378 0.42 0.462))(0.5 0.6 0.7)
A5	(0.333 0.37 0.407) (0.5 0.6 0.7)	(0.225 0.25 0.275)(0.5 0.6 0.7)	(0.522 0.58 0.638))(0.5 0.6 0.7)
A6	(0.432 0.48 0.528)(0.5 0.6 0.7)	(0.549 0.61 0.671))(0.5 0.6 0.7)	(0.621 0.69 0.759))(0.5 0.6 0.7)
A7	(0.738 0.82 0.902)(0.7 0.8 0.9)	(0.324 0.36 0.396))(0.5 0.6 0.7)	(0.522 0.58 0.638))(0.5 0.6 0.7)
A8	(0.531 0.59 0.649)(0.5 0.6 0.7)	(0.378 0.42 0.462) (0.5 0.6 0.7)	(0.648 0.72 0.792)(0.7 0.8 0.9)

Table 9.

Decision matrix

	C_4	C_5
A1	(0.702 0.78 0.858) (0.5 0.6 0.7)	(0.126 0.14 0.154)(0.5 0.6 0.7)
A2	(0.585 0.65 0.715) (0.5 0.6 0.7)	(0.828 0.92 1.012)(0.7 0.8 0.9)
A3	(0.747 0.83 0.913)(0.7 0.8 0.9)	(0.576 0.64 0.704) (0.5 0.6 0.7)
A4	(0.405 0.45 0.495)(0.5 0.6 0.7)	(0.342 0.38 0.418)(0.5 0.6 0.7)
A5	(0.351 0.39 0.429) (0.5 0.6 0.7)	(0.243 0.27 0.297)(0.5 0.6 0.7)
A6	(0.621 0.69 0.759)(0.5 0.6 0.7)	(0.702 0.78 0.858)(0.5 0.6 0.7)
A7	(0.216 0.24 0.264)(0.7 0.8 0.9)	(0.324 0.36 0.396))(0.5 0.6 0.7)
A8	(0.522 0.58 0.638)(0.5 0.6 0.7)	(0.252 0.28 0.308) (0.5 0.6 0.7)

For simplicity, the weights vector of the 5 attributes is given as: weight for C_1 is $W_1 = 0.3$, for C_2 is $W_2 = 0.2$, for C_3 is $W_3 = 0.12$, for C_4 is $W_4 = 0.18$ and for C_5 is $W_5 = 0.2$.

The positive ideal alternative is presented as

$A_p^{id} =$((0.738 0.82 0.902)(0.7 0.8 0.9),(0.711 0.79 0.869)(0.7 0.8 0.9),(0.648 0.72 0.792)(0.7 0.8 0.9),(0.747 0.83 0.913)(0.7 0.8 0.9),(0.828 0.92 1.012)(0.7 0.8 0.9))

The negative ideal alternative is presented as

$A_N^{id} =$((0.126 0.14 0.154)(0.5 0.6 0.7), (0.225 0.25 0.275)(0.5 0.6 0.7), (0.27 0.3 0.33)(0.5 0.6 0.7), (0.216 0.24 0.264)(0.5 0.6 0.7), (0.126 0.14 0.154)(0.5 0.6 0.7))

According to the formulas given above weighted distances between Z-vectors of alternatives and positive ideal solution Z-vector are obtained as

$d_{p1} = 0.32$ $d_{p2} = 0.42$ $d_{p3} = 0.375$ $d_{p4} = 0.24$

$d_{p5} = 0.315$ $d_{p6} = 0.261$ $d_{p7} = 0.246$ $d_{p8} = 0.274$

Analogously we have obtained weighted distances between Z-vectors of alternatives and negative ideal solution Z- vector:

$d_{N1} = 0.18$ $d_{N2} = 0.32$ $d_{N3} = 0.24$ $d_{N4} = 0.27$

$d_{N5} = 0.114$ $d_{N6} = 0.22$ $d_{N7} = 0.429$ $d_{N8} = 0.225$

The membership degree $r_i, i = \overline{1,8}$ are calculated according to (58) and we obtained:

$r_1 = 0.27$ $r_2 = 0.37$ $r_3 = 0.29$ $r_4 = 0.56$ $r_5 = 0.12$ $r_6 = 0.42$ $r_7 = 0.75$ $r_8 = 0.4$

The final decision is determined as

$$\max(r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8) = 0.75$$

The best alternative is A_7 .

Decision making in Project selection by using Z-relation equations

Consider a civil engineering project evaluation problem. Such problems are characterized by a series of criteria. Criteria differ w.r.t. their significance for a project completion. At the same time, each project has different levels of performance w.r.t. criteria. Thus, such problems can be formalized in the realm of multicriteria choice.

In the considered case, three alternative projects a_1, a_2, a_3 are evaluated by using criteria $C_1, ..., C_8$ capital cost, time management, design and structure, environmental impact, quality control, risk management, human factor, and strategic management. Information related to criteria significance and performance of project w.r.t. the criteria is characterized by fuzziness and partial reliability. In view of this criteria significance and projects performance are described by Z-valued constraints (see Tables 10 and 11).

Table 10.

Criteria significance

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Significance	very important, sure	critical, sure	indeed critical, sure	rather important, sure	important, sure	indeed critical, sure	important, sure	important, sure

Table 11.

Criteria evaluations of alternatives

Factor (criterion)	Performance		
	Project 1	Project 2	Project 3
Capital cost	average, sure	indeed superior, sure	poor, sure
time management	indeed superior, sure	below average, sure	superior, sure
design and structure	very poor, sure	indeed superior, sure	average, sure
environmental impact	below average, sure	poor, sure	superior, sure
quality control	superior, sure	average, sure	poor, sure
risk management	poor, sure	below average, sure	indeed superior, sure
human factor	very poor, sure	above average, sure	superior, sure
strategic management	above average, sure	poor, sure	average, sure

The codebooks for Z-number-valued constraints over 0-1 scale are given in Tables 12, and 13.

Table 12.

Codebook for criteria significance

Scale	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
indeed critical, sure	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0.072,0.08,0.088),(0.7,0.8,0.9))	((0.864,0.96,1),(0.7,0.8,0.9))	((0.9,1,1),(0.7,0.8,0.9))
critical, sure	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0.045,0.05,0.055),(0.7,0.8,0.9))	((0.18,0.2,0.22),(0.7,0.8,0.9))	(0.765,0.85,0.935),(0.7,0.8,0.9))	((0.9,1,1),(0.7,0.8,0.9))
very important, sure	((0,0,0),(0.7,0.8,0.9))	((0.009,0.01,0.011),(0.7,0.8,0.9))	((0.081,0.09,0.099),(0.7,0.8,0.9))	((0.441,0.49,0.539),(0.7,0.8,0.9))	(0.729,0.81,0.891),(0.7,0.8,0.9))	((0.9,1,1),(0.7,0.8,0.9))	(0.729,0.81,0.891),(0.7,0.8,0.9))	((0.441,0.49,0.539),(0.7,0.8,0.9))	((0.081,0.09,0.099),(0.7,0.8,0.9))	((0.0099,0.011,0.0121),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
important, sure	((0,0,0),(0.7,0.8,0.9))	((0.09,0.1,0.11),(0.7,0.8,0.9))	((0.27,0.3,0.33),(0.7,0.8,0.9))	((0.63,0.7,0.77),(0.7,0.8,0.9))	((0.81,0.9,0.99),(0.7,0.8,0.9))	((0.9,1,1),(0.7,0.8,0.9))	((0.81,0.9,0.99),(0.7,0.8,0.9))	((0.63,0.7,0.77),(0.7,0.8,0.9))	((0.27,0.3,0.33),(0.7,0.8,0.9))	((0.09,0.1,0.11),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
Rather important, sure	((0,0,0),(0.7,0.8,0.9))	((0.288,0.32,0.352),(0.7,0.8,0.9))	((0.495,0.55,0.605),(0.7,0.8,0.9))	(0.756,0.84,0.924),(0.7,0.8,0.9))	((0.855,0.95,1),(0.7,0.8,0.9))	((0.9,1,1),(0.7,0.8,0.9))	((0.855,0.95,1),(0.7,0.8,0.9))	(0.756,0.84,0.924),(0.7,0.8,0.9))	((0.495,0.55,0.605),(0.7,0.8,0.9))	((0.288,0.32,0.352),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))

Table 13.

Codebook for criteria evaluations (performance)

Scale	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
superior, sure	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0.09,0.1,0.11),(0.7,0.8,0.9))	((0.225,0.25,0.275),(0.7,0.8,0.9))	((0.81,0.9,0.99),(0.7,0.8,0.9))	((0.9,1,1),(0.7,0.8,0.9))
indeed superior, sure	((0,0,0),(0.7,0.8,0.9))	((0.045,0.05,0.055),(0.7,0.8,0.9))	((0.135,0.15,0.165),(0.7,0.8,0.9))	((0.36,0.4,0.44),(0.7,0.8,0.9))	(0.72,0.8,0.88),(0.7,0.8,0.9))	(0.9,1,1),(0.7,0.8,0.9	(0.72,0.8,0.88),(0.7,0.8,0.9))	((0.36,0.4,0.44),(0.7,0.8,0.9))	((0.135,0.15,0.165),(0.7,0.8,0.9))	((0.045,0.05,0.055),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
above average, sure	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0.18,0.2,0.22),(0.7,0.8,0.9))	((0.54,0.6,0.66),(0.7,0.8,0.9))	(0.765,0.85,0.935),(0.7,0.8,0.9))	((0.855,0.95,1),(0.7,0.8,0.9))	(0.9,1,1),(0.7,0.8,0.9
average, sure	((0,0,0),(0.7,0.8,0.9))	((0.045,0.05,0.055),(0.7,0.8,0.9))	((0.135,0.15,0.165),(0.7,0.8,0.9))	((0.36,0.4,0.44),(0.7,0.8,0.9))	(0.72,0.8,0.88),(0.7,0.8,0.9))	(0.9,1,1),(0.7,0.8,0.9))	(0.72,0.8,0.88),(0.7,0.8,0.9))	((0.36,0.4,0.44),(0.7,0.8,0.9))	((0.135,0.15,0.165),(0.7,0.8,0.9))	((0.045,0.05,0.055),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
below average, sure	((0.9,1,1),(0.7,0.8,0.9))	((0.855,0.95,1),(0.7,0.8,0.9))	(0.765,0.85,0.935),(0.7,0.8,0.9))	((0.54,0.6,0.66),(0.7,0.8,0.9))	((0.18,0.2,0.22),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))

very poor, sure	poor, sure
((0.9,1,1),(0.7,0.8,0.9))	((0.9,1,1),(0.7,0.8,0.9))
((0.576,0.64,0.704),(0.7,0.8,0.9))	(0.72,0.8,0.88),(0.7,0.8,0.9))
((0.108,0.12,0.132),(0.7,0.8,0.9))	((0.315,0.35,0.385),(0.7,0.8,0.9))
((0.036,0.04,0.044),(0.7,0.8,0.9))	((0.18,0.2,0.22),(0.7,0.8,0.9))
((0,0,0),(0.7,0.8,0.9))	((0.045,0.05,0.055),(0.7,0.8,0.9))
((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))
((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))

Let us determine the best project. The problem can be treated as a decision problem under Z-valued information described in chapter 4. At the first stage of the solution method, for each pair of a project and a criterion $(a,C), a\in\{a_1,a_2,a_3\}, C\in\{C_1,...,C_8\}$, we found two-dimensional Z-relation $(Z_{(a,C)ij})$ between performance $(Z^{perform}_{(a,C)ij})$ of a project w.r.t. a criterion C and the significance (Z^{signif}_{Cij}) of C by solving the following Z-relation equation:

$$(Z_{(a,C)ij})\circ(Z^{perform}_{(a,C)ij})=(Z^{signif}_{Cij}), \tag{63}$$

where $(Z_{(a,C)ij})$ is found as a maximal solution $(Z_{(a,C)'ij})$:

$$(Z_{(a,C)'ij})=(Z^{perform}_{(a,C)ij})^{-1}\alpha(Z^{signif}_{Cij}). \tag{64}$$

The obtained results are as follows:

Scale	$(Z_{proj\,a_1 i})$	$(Z_{proj\,a_2 i})$	$(Z_{proj\,a_3 i})$
0	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))	((0,0,0),(0.7,0.8,0.9))
0.1	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))	((0.072,0.08,0.088),(0.7,0.8,0.9))
0.2	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))
0.3	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))
0.4	((0.072,0.08,0.088),(0.7,0.8,0.9))	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))
0.5	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((1,1,1),(1,1,1))
0.6	((0,0,0),(0.7,0.8,0.9))	((0,0,0),(0.7,0.8,0.9))	((1,1,1),(1,1,1))
0.7	((0,0,0),(0.7,0.8,0.9))	((0.072,0.08,0.088),(0.7,0.8,0.9))	((1,1,1),(1,1,1))
0.8	((0,0,0),(0.7,0.8,0.9))	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))
0.9	((0,0,0),(0.7,0.8,0.9))	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))
1	((0,0,0),(0.7,0.8,0.9))	((1,1,1),(1,1,1))	((1,1,1),(1,1,1))

Finally, we compared Z-valued overall evaluations $(Z_{proj\,a_1 i}),(Z_{proj\,a_2 i}),(Z_{proj\,a_3 i})$ and obtained the following results: $(Z_{proj\,a_3 i}) \succ (Z_{proj\,a_2 i}) \succ (Z_{proj\,a_1 i})$. Thus, a_3 is the best project.

Results

1. Comprehensive critical analysis of existing decision theories and tools has shown that effective research results on decision making methods under imprecise and partially reliable decision information are very scarce. In particular, up to day in scientific literature there are no works on formulation of more adequate decision preferences, characterized by bimodal information (information containing synergy of fuzzy and probabilistic uncertainty). The concept of Z-number, the Z-extension of fuzzy logic, is a formal construct to describe such kind of information.
2. For synthesis of more adequate decision preferences, we have formulated the concept of Z-number-valued eigenvalue and eigenvector for decision matrices components of which are Z-numbers. A formal statement of the problem and solution method for computation of Z-number-valued eigensolutions were proposed.

3. Decision making is based on preferences over alternatives and choice criteria. The notion of consistency in this case is used to estimate the quality of preference knowledge and its stability for reliable evaluation of decision alternatives. Existing works have a set of strict consistence conditions used to keep the rationality of preference intensities between compared elements. In this study we have proposed an approach to deriving consistency-driven preference for such kind of situation. A preference degree is described by a Z-number to reflect imprecision and partial reliability of preference knowledge. An optimization problem with Z-number-valued variables is used to design consistent preferences.
4. Fuzzy relation equations were main tool for solving a lot of theoretical and practical problems tasks, in particular, decision making problems. Unfortunately, these approaches did not take into account reliability of information. In this dissertation we introduce a definition of Z-relation, and some operations on Z-relations. On this basis, a statement of decision making problem with information described by Z-relations and a solution method were proposed.
5. We have suggested two new methods for multi-attribute decision making problem, where the attribute values are Z-numbers and the weight information on attributes are partially reliable. First method is based on the ideal solution concept and distance between Z-vectors. Final decision alternative is selected on basis of degree of membership of candidates to the ideal solutions. The second method is based on similarity measure. For solving of the considered decision problem, we apply Jaccard similarity measure adopted for Z-information case.
6. Theoretical findings suggested in this dissertation were applied for decision making problem of country selection for business location, hotel business decisions, project selection and web services selection problems. These applications and

comparative analysis of suggested theoretical results show validity and usefulness of the proposed approaches.

The main content of the dissertation is published in the following works:

1. Alizadeh, A.V., Aliev, R.R., Aliyev, R.R. Operational approach to Z-information-based decision making // b-Quadrat Verlag, Germany, - 2012 – p. 269-277.
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Author's individual participation in the published works.

[1, 2, 8, 11, 13] – problem statement, mathematical modeling, computer simulation, and analysis of finding.
[7,9] – reasoning processes and validity testing of the obtained results.

The defense of the dissertation will be held on December 09, 2021 at 1 PM at the meeting of the Dissertation Council ED 2.02 operating under the Azerbaijan State Oil and Industry University.

Address: AZ1010, Baku, Azadlig Avenue 34

The dissertation is available in the library of the Azerbaijan State Oil and Industry University.

Electronic versions of the dissertation and abstract are posted on the official website of the Azerbaijan State Oil and Industry University.

The abstract was sent to the necessary addresses on October 06, 2021.

Signed for publication: 06.10.2021

Paper format: A5

Volume: 39 498

Circulation: 40